Abel Symposia 12

Toke M. Carlsen
Nadia S. Larsen
Sergey Neshveyev
Christian Skau *Editors*

Operator Algebras and Applications

The Abel Symposium 2015

Springer

ABEL SYMPOSIA

Edited by the Norwegian Mathematical Society

More information about this series at http://www.springer.com/series/7462

Participants at the Abel Symposium 2015.
Photo taken by Andrew Toms

Toke M. Carlsen • Nadia S. Larsen •
Sergey Neshveyev • Christian Skau
Editors

Operator Algebras and Applications

The Abel Symposium 2015

ABEL
PRISEN

Springer

Editors
Toke M. Carlsen
Department of Science and Technology
University of the Faroe Islands
Tórshavn, Faroe Islands

Nadia S. Larsen
Department of Mathematics
University of Oslo
Oslo, Norway

Sergey Neshveyev
Department of Mathematics
University of Oslo
Oslo, Norway

Christian Skau
Department of Mathematical Sciences
Norwegian University of Science
and Technology
Trondheim, Norway

ISSN 2193-2808 ISSN 2197-8549 (electronic)
Abel Symposia
ISBN 978-3-319-39284-4 ISBN 978-3-319-39286-8 (eBook)
DOI 10.1007/978-3-319-39286-8

Library of Congress Control Number: 2016945020

Mathematics Subject Classification (2010): 46Lxx, 37Bxx, 19Kxx

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG Switzerland

Foreword

The Norwegian government established the Abel Prize in mathematics in 2002, and the first prize was awarded in 2003. In addition to honoring the great Norwegian mathematician Niels Henrik Abel by awarding an international prize for outstanding scientific work in the field of mathematics, the prize shall contribute toward raising the status of mathematics in society and stimulate the interest for science among school children and students. In keeping with this objective, the Niels Henrik Abel Board has decided to finance annual Abel Symposia. The topic of the symposia may be selected broadly in the area of pure and applied mathematics. The symposia should be at the highest international level and serve to build bridges between the national and international research communities. The Norwegian Mathematical Society is responsible for the events. It has also been decided that the contributions from these symposia should be presented in a series of proceedings, and Springer Verlag has enthusiastically agreed to publish the series. The Niels Henrik Abel Board is confident that the series will be a valuable contribution to the mathematical literature.

Chair of the Niels Henrik Abel Board | Helge Holden

Preface

Målet for vår vitenskap er på den ene side å oppnå nye resultater, og på den annen side å sammenfatte og belyse tidligere resultater sett fra et høyere ståsted.

Sophus Lie[1]

The Abel Symposium 2015 focused on operator algebras and the wide ramifications the field has spawned. Operator algebras form a branch of mathematics that dates back to the work of John von Neumann in the 1930s. Operator algebras were proposed as a framework for quantum mechanics, with the observables replaced by self-adjoint operators on Hilbert spaces and classical algebras of functions replaced by algebras of operators. Spectacular breakthroughs by the Fields medalists Alain Connes and Vaughan Jones marked the beginning of an impressive development, in the course of which operator algebras established important ties with other areas of mathematics, such as geometry, K-theory, number theory, quantum field theory, dynamical systems, and ergodic theory.

The first Abel Symposium, held in 2004, also focused on operator algebras. It is interesting to see the development and the remarkable advances that have been made in this field in the years since, which strikingly illustrate the vitality of the field.

The Abel Symposium 2015 took place on the ship Finnmarken, part of the Coastal Express line (the Norwegian Hurtigruten), which offered a spectacular venue. The ship left Bergen on August 7 and arrived at its final destination, Harstad in the Lofoten Islands, on August 11. The scenery the participants saw on the way north was marvelous; for example, the ship sailed into both the Geirangerfjord and Trollfjord.

There were altogether 26 talks given at the symposium. In keeping with the organizers' goals, there was no single main theme for the symposium, but rather a variety of themes, all highlighting the richness of the subject. It is perhaps appropriate to draw attention to one of the themes of the talks, which is the classification program for nuclear C^*-algebras. In fact, a truly major breakthrough

[1] "The goal of our science is on the one hand to obtain new results, and on the other hand to summarize and illuminate earlier results as seen from a higher vantage point." Sophus Lie

in this area occurred just a few weeks before the Abel Symposium 2015—amazing timing! Some of the protagonists in this effort—one that has stretched over more than 25 years and has involved many researchers—gave talks on this very topic at the symposium. The survey article by Wilhelm Winter in this proceedings volume offers a panoramic view of the developments in the classification program leading up to the breakthrough mentioned above.

Alain Connes and Vaughan Jones were also among the participants, and they gave talks on topics ranging, respectively, from gravity and the standard model in physics to subfactors, knot theory, and the Thompson group, thus illustrating the broad ramifications of operator algebras in modern mathematics.

Ola Bratteli and Uffe Haagerup, two main contributors to the theory of operator algebras, tragically passed away in the months before the symposium. Their legacy was commemorated and honored in a talk by Erling Størmer. One of the articles in this volume is by Uffe Haagerup, and its publication was made possible with the help of three of Haagerup's colleagues from the University of Copenhagen, to whom he had privately communicated the results shortly before his untimely passing.

The articles in this volume are organized alphabetically rather than thematically. Some are research articles that present new results, others are surveys that cover the development of a specific line of research, and yet others offer a combination of survey and research. These contributions offer a multifaceted portrait of beautiful mathematics that both newcomers to the field of operator algebras and seasoned researchers alike will appreciate.

Tórshavn, Faroe Islands — Toke M. Carlsen
Oslo, Norway — Nadia S. Larsen
Oslo, Norway — Sergey Neshveyev
Trondheim, Norway — Christian Skau
April 2016

Contents

C*-Tensor Categories and Subfactors for Totally Disconnected Groups

Yuki Arano and Stefaan Vaes

Abstract We associate a rigid C*-tensor category $\mathscr{C}$ to a totally disconnected locally compact group G and a compact open subgroup $K < G$. We characterize when $\mathscr{C}$ has the Haagerup property or property (T), and when $\mathscr{C}$ is weakly amenable. When G is compactly generated, we prove that $\mathscr{C}$ is essentially equivalent to the planar algebra associated by Jones and Burstein to a group acting on a locally finite bipartite graph. We then concretely realize $\mathscr{C}$ as the category of bimodules generated by a hyperfinite subfactor.

1 Introduction

Rigid C*-tensor categories arise as representation categories of compact groups and compact quantum groups and also as (part of) the standard invariant of a finite index subfactor. They can be viewed as a discrete group like structure and this analogy has lead to a lot of recent results with a flavor of geometric group theory, see [9, 17, 18, 25, 26].

In this paper, we introduce a rigid C*-tensor category $\mathscr{C}$ canonically associated with a totally disconnected locally compact group G and a compact open subgroup $K < G$. Up to Morita equivalence, $\mathscr{C}$ does not depend on the choice of K. The tensor category $\mathscr{C}$ can be described in several equivalent ways, see Sect. 2. Here, we mention that the representation category of K is a full subcategory of $\mathscr{C}$ and that the "quotient" of the fusion algebra of $\mathscr{C}$ by Rep K is the Hecke algebra of finitely supported functions on $K\backslash G/K$ equipped with the convolution product.

When G is compactly generated, we explain how the C*-tensor category $\mathscr{C}$ is related to the planar algebra $\mathscr{P}$ (i.e. standard invariant of a subfactor) associated in [5, 11] with a locally finite bipartite graph $\mathscr{G}$ and a closed subgroup $G < \mathrm{Aut}(\mathscr{G})$. At

Y. Arano
Graduate School of Mathematics, University of Tokyo, 3-8-1 Komaba, Meguro-ku, Tokyo, Japan
e-mail: arano@ms.u-tokyo.ac.jp

S. Vaes (✉)
KU Leuven, Department of Mathematics, Leuven, Belgium
e-mail: stefaan.vaes@wis.kuleuven.be

T.M. Carlsen et al. (eds.), *Operator Algebras and Applications*, Abel Symposia 12,
DOI 10.1007/978-3-319-39286-8_1

the same time, we prove that these planar algebras $\mathscr{P}$ can be realized by a *hyperfinite* subfactor.

Given a finite index subfactor $N \subset M$, the notions of *amenability*, *Haagerup property* and *property (T)* for its standard invariant $\mathscr{G}_{N,M}$ were introduced by Popa in [23, 24] in terms of the associated symmetric enveloping algebra $T \subset S$ (see [21, 23]) and shown to only depend on $\mathscr{G}_{N,M}$. Denoting by $\mathscr{C}$ the tensor category of M-M-bimodules generated by the subfactor, these properties were then formulated in [26] intrinsically in terms of $\mathscr{C}$, and in particular directly in terms of $\mathscr{G}_{N,M}$. We recall these definitions and equivalent formulations in Sect. 4. Similarly, *weak amenability* and the corresponding Cowling-Haagerup constant for the standard invariant $\mathscr{G}_{N,M}$ of a subfactor $N \subset M$ were first defined in terms of the symmetric enveloping inclusion in [3] and then intrinsically for rigid C*-tensor categories in [26], see Sect. 5. Reinterpreting [1, 6], it was proved in [26] that the representation category of $\mathrm{SU}_q(2)$ (and thus, the Temperley-Lieb-Jones standard invariant) is weakly amenable and has the Haagerup property, while the representation category of $\mathrm{SU}_q(3)$ has property (T).

For the C*-tensor categories $\mathscr{C}$ that we associate to a totally disconnected group G, we characterize when $\mathscr{C}$ has the Haagerup property or property (T) and when $\mathscr{C}$ is weakly amenable. We give several examples and counterexamples, in particular illustrating that the Haagerup property/weak amenability of G is not sufficient for $\mathscr{C}$ to have the Haagerup property or to be weakly amenable. Even more so, when $\mathscr{C}$ is the category associated with $G = \mathrm{SL}(2, \mathbb{Q}_p)$, then the subcategory $\operatorname{Rep} K$ with $K = \mathrm{SL}(2, \mathbb{Z}_p)$ has the relative property (T). When $G = \mathrm{SL}(n, \mathbb{Q}_p)$ with $n \geq 3$, the tensor category $\mathscr{C}$ has property (T), but we also give examples of property (T) groups G such that $\mathscr{C}$ does not have property (T).

Our main technical tool is Ocneanu's tube algebra [19] associated with any rigid C*-tensor category, see Sect. 3. When $\mathscr{C}$ is the C*-tensor category of a totally disconnected group G, we prove that the tube algebra is isomorphic with a canonical dense $*$-subalgebra of $C_0(G) \rtimes_{\mathrm{Ad}} G$, where G acts on G by conjugation. We can therefore express the above mentioned approximation and rigidity properties of the tensor category $\mathscr{C}$ in terms of G and the dynamics of the action $G \curvearrowright^{\mathrm{Ad}} G$ by conjugation.

In this paper, all locally compact groups are assumed to be second countable. We call totally disconnected group every second countable, locally compact, totally disconnected group.

2 C*-Tensor Categories of Totally Disconnected Groups

Throughout this section, fix a totally disconnected group G. For all compact open subgroups $K_1, K_2 < G$, we define

$\mathscr{C}_1$: the category of K_1-K_2-$L^\infty(G)$-modules, i.e. Hilbert spaces $\mathscr{H}$ equipped with commuting unitary representations $(\lambda(k_1))_{k_1 \in K_1}$ and $(\rho(k_2))_{k_2 \in K_2}$ and with

a normal $*$-representation $\Pi : L^\infty(G) \to B(\mathscr{H})$ that are equivariant with respect to the left translation action $K_1 \curvearrowright G$ and the right translation action $K_2 \curvearrowright G$;

$\mathscr{C}_2$: the category of K_1-$L^\infty(G/K_2)$-modules, i.e. Hilbert spaces $\mathscr{H}$ equipped with a unitary representation $(\pi(k_1))_{k_1 \in K_1}$ and a normal $*$-representation $\Pi : L^\infty(G/K_2) \to B(\mathscr{H})$ that are covariant with respect to the left translation action $K_1 \curvearrowright G/K_2$;

$\mathscr{C}_3$: the category of G-$L^\infty(G/K_1)$-$L^\infty(G/K_2)$-modules, i.e. Hilbert spaces $\mathscr{H}$ equipped with a unitary representation $(\pi(g))_{g\in G}$ and with an $L^\infty(G/K_1)$-$L^\infty(G/K_2)$-bimodule structure that are equivariant with respect to the left translation action of G on G/K_1 and G/K_2;

and with morphisms given by bounded operators that intertwine the given structure.

Let $K_3 < G$ also be a compact open subgroup. We define the tensor product $\mathscr{H} \otimes_{K_2} \mathscr{K}$ of a K_1-K_2-$L^\infty(G)$-module $\mathscr{H}$ and a K_2-K_3-$L^\infty(G)$-module $\mathscr{K}$ as the Hilbert space

$$\mathscr{H} \otimes_{K_2} \mathscr{K} = \{\xi \in \mathscr{H} \otimes \mathscr{K} \mid (\rho(k_2) \otimes \lambda(k_2))\xi = \xi \ \text{ for all } \ k_2 \in K_2\}$$

equipped with the unitary representations $(\lambda(k_1) \otimes 1)_{k_1 \in K_1}$ and $(1 \otimes \rho(k_3))_{k_3\in K_3}$ and with the representation $(\Pi_{\mathscr{H}} \otimes \Pi_{\mathscr{K}}) \circ \Delta$ of $L^\infty(G)$, where we denote by $\Delta : L^\infty(G) \to L^\infty(G) \mathbin{\overline{\otimes}} L^\infty(G)$ the comultiplication given by $(\Delta(F))(g,h) = F(gh)$ for all $g, h \in G$.

The tensor product of a G-$L^\infty(G/K_1)$-$L^\infty(G/K_2)$-module $\mathscr{H}$ and a G-$L^\infty(G/K_2)$-$L^\infty(G/K_3)$-module $\mathscr{K}$ is denoted as $\mathscr{H} \otimes_{L^\infty(G/K_2)} \mathscr{K}$ and defined as the Hilbert space

$$\begin{aligned}
&\mathscr{H} \otimes_{L^\infty(G/K_2)} \mathscr{K} \\
&= \{\xi \in \mathscr{H} \otimes \mathscr{K} \mid \xi(1_{gK_2} \otimes 1) = (1 \otimes 1_{gK_2})\xi \ \text{ for all } \ gK_2 \in G/K_2\} \\
&= \bigoplus_{g \in G/K_2} \mathscr{H} \cdot 1_{gK_2} \otimes 1_{gK_2} \cdot \mathscr{K}
\end{aligned}$$

with the unitary representation $(\pi_{\mathscr{H}}(g) \otimes \pi_{\mathscr{K}}(g))_{g\in G}$ and with the $L^\infty(G/K_1)$-$L^\infty(G/K_3)$-bimodule structure given by the left action of $1_{gK_1} \otimes 1$ for $gK_1 \in G/K_1$ and the right action of $1 \otimes 1_{hK_3}$ for $hK_3 \in G/K_3$.

We say that objects $\mathscr{H}$ are of *finite rank*

$\mathscr{C}_1$: if $\mathscr{H}_{K_2} := \{\xi \in \mathscr{H} \mid \rho(k_2)\xi = \xi \ \text{ for all } \ k_2 \in K_2\}$ is finite dimensional; as we will see in the proof of Proposition 2.2, this is equivalent with requiring that ${}_{K_1}\mathscr{H}$ is finite dimensional;

$\mathscr{C}_2$: if $\mathscr{H}$ is finite dimensional;

$\mathscr{C}_3$: if $1_{eK_1} \cdot \mathscr{H}$ is finite dimensional; as we will see in the proof of Proposition 2.2, this is equivalent with requiring that $\mathscr{H} \cdot 1_{eK_2}$ is finite dimensional.

Altogether, we get that $\mathscr{C}_1$ and $\mathscr{C}_3$ are *C^*-2-categories*. In both cases, the 0-cells are the compact open subgroups of G. For all compact open subgroups $K_1, K_2 < G$, the 1-cells are the categories $\mathscr{C}_i(K_1, K_2)$ defined above and $\mathscr{C}_i(K_1, K_2) \times \mathscr{C}_i(K_2, K_3) \to \mathscr{C}_i(K_1, K_3)$ is given by the tensor product operation that we just introduced. Restricting to finite rank objects, we get rigid C*-2-categories.

Another typical example of a C*-2-category is given by Hilbert bimodules over II_1 factors: the 0-cells are II_1 factors, the 1-cells are the categories $\mathrm{Bimod}_{M_1\text{-}M_2}$ of Hilbert M_1-M_2-bimodules and $\mathrm{Bimod}_{M_1\text{-}M_2} \times \mathrm{Bimod}_{M_2\text{-}M_3} \to \mathrm{Bimod}_{M_1\text{-}M_3}$ is given by the Connes tensor product. Again, restricting to finite index bimodules, we get a rigid C*-2-category.

Remark 2.1 The standard invariant of an extremal finite index subfactor $N \subset M$ can be viewed as follows as a rigid C*-2-category. There are only two 0-cells, namely N and M; the 1-cells are the N-N, N-M, M-N and M-M-bimodules generated by the subfactor; and we are given a favorite and generating 1-cell from N to M, namely the N-M-bimodule $L^2(M)$.

Abstractly, a rigid C*-2-category $\mathscr{C}$ with only two 0-cells (say $+$ and $-$), irreducible tensor units in $\mathscr{C}_{++}$ and $\mathscr{C}_{--}$, and a given generating object $\mathscr{H} \in \mathscr{C}_{+-}$ is exactly the same as a standard λ-lattice in the sense of Popa [22, Definitions 1.1 and 2.1]. Indeed, for every $n \geq 0$, define $\mathscr{H}_{+,n}$ as the n-fold alternating tensor product of $\mathscr{H}$ and $\overline{\mathscr{H}}$ starting with $\mathscr{H}$. Similarly, define $\mathscr{H}_{-,n}$ by starting with $\overline{\mathscr{H}}$. For $0 \leq j$, define $A_{0j} = \mathrm{End}(\mathscr{H}_{+,j})$. When $0 \leq i \leq j < \infty$, define $A_{ij} \subset A_{0j}$ as $A_{ij} := 1^i \otimes \mathrm{End}(\mathscr{H}_{(-1)^i,j-i})$ viewed as a subalgebra of $A_{0j} = \mathrm{End}(\mathscr{H}_{+,j})$ by writing $\mathscr{H}_{+,j} = \mathscr{H}_{+,i}\mathscr{H}_{(-1)^i,j-i}$. The standard solutions for the conjugate equations (see Sect. 3) give rise to canonical projections $e_+ \in \mathrm{End}(\mathscr{H}\overline{\mathscr{H}})$ and $e_- \in \mathrm{End}(\overline{\mathscr{H}}\mathscr{H})$ given by

$$e_+ = d(\mathscr{H})^{-1} s_{\mathscr{H}} s_{\mathscr{H}}^* \quad \text{and} \quad e_- = d(\mathscr{H})^{-1} t_{\mathscr{H}} t_{\mathscr{H}}^* ,$$

and thus to a representation of the Jones projections $e_j \in A_{kl}$ (for $k < j < l$). Finally, if we equip all A_{ij} with the normalized categorical trace, we have defined a standard λ-lattice in the sense of [22, Definitions 1.1 and 2.1]. Given two rigid C*-2-categories with fixed generating objects as above, it is straightforward to check that the associated standard λ-lattices are isomorphic if and only if there exists an equivalence of C*-2-categories preserving the generators. Conversely given a standard λ-lattice $\mathscr{G}$, by [22, Theorem 3.1], there exists an extremal subfactor $N \subset M$ whose standard invariant is $\mathscr{G}$ and we can define $\mathscr{C}$ as the C*-2-category of the subfactor $N \subset M$, generated by the N-M-bimodule $L^2(M)$ as in the beginning of this remark. One can also define $\mathscr{C}$ directly in terms of $\mathscr{G}$ (see e.g. [14, Section 4.1] for a planar algebra version of this construction).

Thus, also subfactor planar algebras in the sense of [12] are "the same" as rigid C*-2-categories with two 0-cells and such a given generating object $\mathscr{H} \in \mathscr{C}_{+-}$.

For more background on rigid C*-tensor categories, we refer to [16].

Proposition 2.2 *The* C^**-2-categories* $\mathscr{C}_1$ *and* $\mathscr{C}_3$ *are naturally equivalent. In particular, fixing* $K_1 = K_2 = K$*, we get the naturally equivalent rigid* C^**-tensor categories* $\mathscr{C}_{1,f}(K < G)$ *and* $\mathscr{C}_{3,f}(K < G)$*. Up to Morita equivalence,*[1] *these do not depend on the choice of compact open subgroup* $K < G$*.*

Proof Using the left and right translation operators λ_g and ρ_g on $L^2(G)$, one checks that the following formulae define natural equivalences and their inverses between the categories $\mathscr{C}_1$, $\mathscr{C}_2$ and $\mathscr{C}_3$.

- $\mathscr{C}_1 \to \mathscr{C}_2 : \mathscr{H} \mapsto \mathscr{H}_{K_2}$, where $\mathscr{H}_{K_2}$ is the space of right K_2-invariant vectors and where the K_1-$L^\infty(G/K_2)$-module structure on $\mathscr{H}_{K_2}$ is given by restricting the corresponding structure on $\mathscr{H}$.
- $\mathscr{C}_2 \to \mathscr{C}_1 : \mathscr{H} \mapsto \mathscr{H} \otimes_{L^\infty(G/K_2)} L^2(G)$ given by

$$\{\xi \in \mathscr{H} \otimes L^2(G) \mid (1_{gK_2} \otimes 1)\xi = (1 \otimes 1_{gK_2})\xi \;\text{ for all }\; g \in G\}$$
$$= \bigoplus_{g \in G/K_2} 1_{gK_2} \cdot \mathscr{H} \otimes L^2(gK_2)$$

  and where the K_1-K_2-$L^\infty(G)$-module structure is given by $(\lambda_{\mathscr{H}}(k_1) \otimes \lambda_{k_1})_{k_1 \in K_1}$, $(1 \otimes \rho_{k_2})_{k_2 \in K_2}$ and multiplication with $1 \otimes F$ when $F \in L^\infty(G)$.
- $\mathscr{C}_3 \to \mathscr{C}_2 : \mathscr{H} \mapsto 1_{eK_1} \cdot \mathscr{H}$ and where the K_1-$L^\infty(G/K_2)$-module structure on $1_{eK_1} \cdot \mathscr{H}$ is given by restricting the corresponding structure on $\mathscr{H}$.
- $\mathscr{C}_2 \to \mathscr{C}_3 : \mathscr{H} \mapsto L^2(G) \otimes_{K_1} \mathscr{H}$ given by

$$\{\xi \in L^2(G) \otimes \mathscr{H} \mid (\rho_{k_1} \otimes \pi(k_1))\xi = \xi \;\text{ for all }\; k_1 \in K_1\}$$

  and where the G-$L^\infty(G/K_1)$-$L^\infty(G/K_2)$-module structure is given by the representation $(\lambda_g \otimes 1)_{g \in G}$, multiplication with $F \otimes 1$ for $F \in L^\infty(G/K_1)$ and multiplication with $(\mathrm{id} \otimes \Pi)\Delta(F)$ for $F \in L^\infty(G/K_2)$.

By definition, if $\mathscr{H} \in \mathscr{C}_1$ has finite rank, the Hilbert space $\mathscr{H}_{K_2}$ is finite dimensional. Conversely, if $\mathscr{K} \in \mathscr{C}_2$ and $\mathscr{K}$ is a finite dimensional Hilbert space, then the corresponding object $\mathscr{H} \in \mathscr{C}_1$ has the property that both ${}_{K_1}\mathscr{H}$ and $\mathscr{H}_{K_2}$ are finite dimensional. Therefore, $\mathscr{H} \in \mathscr{C}_1$ has finite rank if and only if ${}_{K_1}\mathscr{H}$ is a finite dimensional Hilbert space. A similar reasoning holds for objects in $\mathscr{C}_3$.

It is straightforward to check that the resulting equivalence $\mathscr{C}_1 \leftrightarrow \mathscr{C}_3$ preserves tensor products, so that we have indeed an equivalence between the C*-2-categories $\mathscr{C}_1$ and $\mathscr{C}_3$.

To prove the final statement in the proposition, it suffices to observe that for all compact open subgroups $K_1, K_2 < G$, we have that $L^2(K_1K_2)$ is a nonzero finite rank K_1-K_2-$L^\infty(G)$-module and that $L^2(G/(K_1 \cap K_2))$ is a nonzero finite rank

[1] In the sense of [15, Section 4], where the terminology weak Morita equivalence is used; see also [25, Definition 7.3] and [18, Section 3].

G-$L^\infty(G/K_1)$-$L^\infty(G/K_2)$-module, so that $\mathscr{C}_{i,f}(K_1 < G)$ and $\mathscr{C}_{i,f}(K_2 < G)$ are Morita equivalent for $i = 1, 3$. □

The rigid C*-2-categories $\mathscr{C}_1$ and $\mathscr{C}_2$ can as follows be fully faithfully embedded in the category of bimodules over the hyperfinite II_1 factor. We construct this embedding in an extremal way in the sense of subfactors (cf. Corollary 2.4).

To do so, given a totally disconnected group G, we fix a continuous action $G \curvearrowright^\alpha P$ of G on the hyperfinite II_∞ factor P that is *strictly outer* in the sense of [27, Definition 2.1]: the relative commutant $P' \cap P \rtimes G$ equals $\mathbb{C}1$. Moreover, we should choose this action in such a way that $\mathrm{Tr} \circ \alpha_g = \Delta(g)^{-1/2}\,\mathrm{Tr}$ for all $g \in G$ (where Δ is the modular function on G) and such that there exists a projection $p \in P$ of finite trace with the property that $\alpha_k(p) = p$ whenever k belongs to a compact subgroup of G. Such an action indeed exists: write $P = R_0 \overline{\otimes} R_1$ where R_0 is a copy of the hyperfinite II_1 factor and R_1 is a copy of the hyperfinite II_∞ factor. Choose a continuous trace scaling action $\mathbb{R}_0^+ \curvearrowright^{\alpha_1} R_1$. By [27, Corollary 5.2], we can choose a strictly outer action $G \curvearrowright^{\alpha_0} R_0$. We then define $\alpha_g = (\alpha_0)_g \otimes (\alpha_1)_{\Delta(g)^{-1/2}}$ and we take $p = 1 \otimes p_1$, where $p_1 \in R_1$ is any projection of finite trace. Whenever k belongs to a compact subgroup of G, we have $\Delta(k) = 1$ and thus $\alpha_k(p) = p$.

Whenever $K_1, K_2 < G$ are compact open subgroups of G, we write

$$[K_1 : K_2] = [K_1 : K_1 \cap K_2]\,[K_2 : K_1 \cap K_2]^{-1} \; .$$

Fixing a left Haar measure λ on G, we have $[K_1 : K_2] = \lambda(K_1)\,\lambda(K_2)^{-1}$. Therefore, we have that $[K : gKg^{-1}] = \Delta(g)$ for all compact open subgroups $K < G$ and all $g \in G$.

Theorem 2.3 *Let G be a totally disconnected group and choose a strictly outer action $G \curvearrowright^\alpha P$ on the hyperfinite II_∞ factor P and a projection $p \in P$ as above. For every compact open subgroup $K < G$, write $R(K) = (pPp)^K$. Then each $R(K)$ is a copy of the hyperfinite II_1 factor.*

To every K_1-K_2-$L^\infty(G)$-module $\mathscr{H}$, we associate the Hilbert $R(K_1)$-$R(K_2)$-bimodule $\mathscr{K}$ given by (2.1) below. Then $\mathscr{H} \mapsto \mathscr{K}$ is a fully faithful 2-functor. Also, $\mathscr{H}$ has finite rank if and only if $\mathscr{K}$ is a finite index bimodule. In that case,

$$\dim_{R(K_1)-}(\mathscr{K}) = [K_1 : K_2]^{1/2}\;\dim_{\mathscr{C}_1}(\mathscr{H}) \quad \textit{and}$$

$$\dim_{-R(K_2)}(\mathscr{K}) = [K_2 : K_1]^{1/2}\;\dim_{\mathscr{C}_1}(\mathscr{H}) \; ,$$

where $\dim_{\mathscr{C}_1}(\mathscr{H})$ is the categorical dimension of $\mathscr{H} \in \mathscr{C}_1$.

Proof Given a K_1-K_2-$L^\infty(G)$-module $\mathscr{H}$, turn $\mathscr{H} \otimes L^2(P)$ into a Hilbert $(P \rtimes K_1)$-$(P \rtimes K_2)$-bimodule via

$$u_k \cdot (\xi \otimes b) \cdot u_r = \lambda(k)\rho(r)^*\xi \otimes \alpha_r^{-1}(b) \quad \text{for all } \; k \in K_1, r \in K_2, \xi \in \mathscr{H}, b \in L^2(P),$$

$$a \cdot \zeta \cdot d = (\Pi \otimes \mathrm{id})\alpha(a)\,\zeta\,(1 \otimes d) \qquad \text{for all } \; a, d \in P, \zeta \in \mathscr{H} \otimes L^2(P),$$

where $\alpha : P \to L^\infty(G) \overline{\otimes} P$ is given by $(\alpha(a))(g) = \alpha_g^{-1}(a)$.

Whenever $K < G$ is a compact open subgroup, we define the projection $p_K \in L(G)$ given by

$$p_K = \lambda(K)^{-1} \int_K \lambda_k \, dk \; .$$

We also write $e_K = pp_K$ viewed as a projection in $P \rtimes K$. Since $P \subset P \rtimes K \subset P \rtimes G$, we have that $P' \cap (P \rtimes K) = \mathbb{C}1$, so that $P \rtimes K$ is a factor. So, $P \rtimes K$ is a copy of the hyperfinite II_∞ factor and $e_K \in P \rtimes K$ is a projection of finite trace. We identify $R(K) = e_K(P \rtimes K)e_K$ through the bijective $*$-isomorphism $(pPp)^K \to e_K(P \rtimes K)e_K : a \mapsto ap_K$. In particular, $R(K)$ is a copy of the hyperfinite II_1 factor.

So, for every K_1-K_2-$L^\infty(G)$-module $\mathscr{H}$, we can define the $R(K_1)$-$R(K_2)$-bimodule

$$\mathscr{K} = e_{K_1} \cdot (\mathscr{H} \otimes L^2(P)) \cdot e_{K_2} \; . \tag{2.1}$$

We claim that $\mathrm{End}_{R(K_1)-R(K_2)}(\mathscr{K}) = \mathrm{End}_{\mathscr{C}_1}(\mathscr{H})$ naturally. More concretely, we have to prove that

$$\mathrm{End}_{(P \rtimes K_1)-(P \rtimes K_2)}(\mathscr{H} \otimes L^2(P)) = \mathrm{End}_{\mathscr{C}_1}(\mathscr{H}) \otimes 1 \; , \tag{2.2}$$

where $\mathrm{End}_{\mathscr{C}_1}(\mathscr{H})$ consists of all bounded operators on $\mathscr{H}$ that commute with $\lambda(K_1)$, $\rho(K_2)$ and $\Pi(L^\infty(G))$. To prove (2.2), it is sufficient to show that

$$\mathrm{End}_{P-P}(\mathscr{H} \otimes L^2(P)) = \Pi(L^\infty(G))' \otimes 1 \; . \tag{2.3}$$

Note that the left hand side of (2.3) equals $(\Pi \otimes \mathrm{id})\alpha(P)' \cap B(\mathscr{H}) \mathbin{\overline{\otimes}} P$. Assume that $T \in (\Pi \otimes \mathrm{id})\alpha(P)' \cap B(\mathscr{H}) \mathbin{\overline{\otimes}} P$. In the same was as in [27, Proposition 2.7], it follows that $T \in \Pi(L^\infty(G))' \cap 1$. For completeness, we provide a detailed argument. Define the unitary $W \in L^\infty(G) \mathbin{\overline{\otimes}} L(G)$ given by $W(g) = \lambda_g$. We view both T and $(\Pi \otimes \mathrm{id})(W)$ as elements in $B(\mathscr{H}) \mathbin{\overline{\otimes}} (P \rtimes G)$. For all $a \in P$, we have

$$\begin{aligned}(\Pi \otimes \mathrm{id})(W)\, T\, (\Pi \otimes \mathrm{id})(W)^* \,(1 \otimes a) &= (\Pi \otimes \mathrm{id})(W)\, T\, (\Pi \otimes \mathrm{id})\alpha(a)\, (\Pi \otimes \mathrm{id})(W)^* \\ &= (1 \otimes a)\, (\Pi \otimes \mathrm{id})(W)\, T\, (\Pi \otimes \mathrm{id})(W)^* \; .\end{aligned}$$

Since the action α is strictly outer, we conclude that $(\Pi \otimes \mathrm{id})(W)\, T\, (\Pi \otimes \mathrm{id})(W)^* = S \otimes 1$ for some $S \in B(\mathscr{H})$. So,

$$T = (\Pi \otimes \mathrm{id})(W)^* \,(S \otimes 1)\, (\Pi \otimes \mathrm{id})(W) \; .$$

The left hand side belongs to $B(\mathscr{H}) \mathbin{\overline{\otimes}} P$, while the right hand side belongs to $B(\mathscr{H}) \otimes L(G)$, and both are viewed inside $B(\mathscr{H}) \mathbin{\overline{\otimes}} (P \rtimes G)$. Since $P \cap L(G) = \mathbb{C}1$,

we conclude that $T = T_0 \otimes 1$ for some $T_0 \in B(\mathscr{H})$ and that

$$T_0 \otimes 1 = (\Pi \otimes \mathrm{id})(W)^* \, (S \otimes 1) \, (\Pi \otimes \mathrm{id})(W) \; .$$

Defining the normal $*$-homomorphism $\Psi : L(G) \to L(G) \overline{\otimes} L(G)$ given by $\Psi(\lambda_g) = \lambda_g \otimes \lambda_g$ for all $g \in G$, we apply $\mathrm{id} \otimes \Psi$ and conclude that

$$\begin{aligned} T_0 \otimes 1 \otimes 1 &= (\Pi \otimes \mathrm{id})(W)^*_{13} \, (\Pi \otimes \mathrm{id})(W)^*_{12} \, (S \otimes 1) \, (\Pi \otimes \mathrm{id})(W)_{12} \, (\Pi \otimes \mathrm{id})(W)_{13} \\ &= (\Pi \otimes \mathrm{id})(W)^*_{13} \, (T_0 \otimes 1 \otimes 1) \, (\Pi \otimes \mathrm{id})(W)_{13} \; . \end{aligned}$$

It follows that T_0 commutes with $\Pi(L^\infty(G))$ and (2.2) is proven.

It is easy to check that $\mathscr{H} \mapsto \mathscr{K}$ naturally preserves tensor products. So, we have found a fully faithful 2-functor from $\mathscr{C}_1$ to the C*-2-category of Hilbert bimodules over hyperfinite II$_1$ factors.

To compute $\dim_{-R(K_2)}(\mathscr{K})$, observe that for all $k \in K_1$, $r \in K_2$ and $g \in G$, we have $\alpha_{kgr}(p) = \alpha_{kg}(p) = \alpha_g(\alpha_{g^{-1}kg}(p)) = \alpha_g(p)$. Therefore, as a right $(P \rtimes K_2)$-module, we have

$$e_{K_1} \cdot (\mathscr{H} \otimes L^2(P)) \cong \bigoplus_{g \in K_1 \backslash G / K_2} \big(\mathscr{L}_g \otimes L^2(p_g P) \big) \; ,$$

where $p_g = \alpha_g^{-1}(p)$, where the Hilbert space $\mathscr{L}_g := \Pi(1_{K_1 g K_2})({}_{K_1}\mathscr{H})$ comes with the unitary representation $(\rho(r))_{r \in K_2}$ and where the right $(P \rtimes K_2)$-module structure on $\mathscr{L}_g \otimes L^2(p_g P)$ is given by

$$(\xi \otimes b) \cdot (d u_r) = \rho(r)^* \xi \otimes \alpha_r^{-1}(bd) \quad \text{for all} \;\; \xi \in \mathscr{L}_g, b \in L^2(p_g P), d \in P, r \in K_2 \; .$$

Since $p_g P p_g \rtimes K_2 = p_g (P \rtimes K_2) p_g$ is a factor (actually, $K_2 \curvearrowright p_g P p_g$ is a so-called minimal action), it follows from [28, Theorem 12] that there exists a unitary $V_g \in B(\mathscr{L}_g) \overline{\otimes} p_g P p_g$ satisfying

$$(\mathrm{id} \otimes \alpha_r)(V_g) = V_g(\rho(r) \otimes 1) \quad \text{for all} \;\; r \in K_2 \; .$$

Then left multiplication with V_g intertwines the right $(P \rtimes K_2)$-module structure on the Hilbert space $\mathscr{L}_g \otimes L^2(p_g P)$ with the right $(P \rtimes K_2)$-module structure given by

$$(\xi \otimes b) \cdot (d u_r) = \xi \otimes \alpha_r^{-1}(bd) \quad \text{for all} \;\; \xi \in \mathscr{L}_g, b \in L^2(p_g P), d \in P, r \in K_2 \; .$$

Therefore,

$$\begin{aligned} \dim_{-R(K_2)}\big(\mathscr{L}_g \otimes L^2(p_g P)\big) \cdot e_{K_2} &= \dim(\mathscr{L}_g) \; \dim_{-(pPp)^{K_2}}\big(L^2(p_g P^{K_2} p)\big) \\ &= \dim(\mathscr{L}_g) \, \frac{\mathrm{Tr}(p_g)}{\mathrm{Tr}(p)} = \dim(\mathscr{L}_g) \, \Delta(g)^{1/2} \; . \end{aligned}$$

So, we have proved that

$$\dim_{-R(K_2)}(\mathscr{K}) = \sum_{g \in K_1 \backslash G/K_2} \dim\bigl(\Pi(1_{K_1 g K_2})({}_{K_1}\mathscr{H})\bigr)\ \Delta(g)^{1/2} .$$

We similarly get that

$$\dim_{R(K_1)-}(\mathscr{K}) = \sum_{g \in K_1 \backslash G/K_2} \dim\bigl(\Pi(1_{K_1 g K_2})(\mathscr{H}_{K_2})\bigr)\ \Delta(g)^{-1/2} .$$

To make the connection with the categorical dimension of $\mathscr{H}$, it is useful to view $\mathscr{H}$ as the image of a G-$L^\infty(G/K_1)$-$L^\infty(G/K_2)$-module $\mathscr{H}'$ under the equivalence of Proposition 2.2. This means that we can view $\mathscr{H}$ as the space of L^2-functions $\xi : G \to \mathscr{H}'$ with the property that $\xi(g) \in 1_{eK_1} \cdot \mathscr{H}' \cdot 1_{gK_2}$ for a.e. $g \in G$. The $L^\infty(G)$-module structure of $\mathscr{H}$ is given by pointwise multiplication, while the K_1-K_2-module structure on $\mathscr{H}$ is given by

$$(k \cdot \xi \cdot r)(g) = \pi(k)\xi(k^{-1} g r^{-1}) \quad \text{for all } \ k \in K_1, r \in K_2, g \in G .$$

With this picture, it is easy to see that

$$\Pi(1_{K_1 g K_2})(\mathscr{H}_{K_2}) \cong 1_{eK_1} \cdot \mathscr{H}' \cdot 1_{K_1 g K_2} .$$

The map $\xi \mapsto \tilde{\xi}$ with $\tilde{\xi}(g) = \pi(g)^* \xi(g)$ is an isomorphism between $\mathscr{H}$ and the space of L^2-functions $\eta : G \to \mathscr{H}'$ with the property that $\eta(g) \in 1_{g^{-1}K_1} \cdot \mathscr{H}' \cdot 1_{eK_2}$ for a.e. $g \in G$. The $L^\infty(G)$-module structure is still given by pointwise multiplication, while the K_1-K_2-module structure is now given by

$$(k \cdot \eta \cdot r)(g) = \pi(r)^* \eta(k^{-1} g r^{-1}) .$$

In this way, we get that

$$\Pi(1_{K_1 g K_2})({}_{K_1}\mathscr{H}) \cong 1_{K_2 g^{-1} K_1} \cdot \mathscr{H}' \cdot 1_{eK_2} .$$

It thus follows that

$$\dim_{-R(K_2)}(\mathscr{K}) = \sum_{g \in K_1 \backslash G/K_2} \dim(1_{K_2 g^{-1} K_1} \cdot \mathscr{H}' \cdot 1_{eK_2})\, \Delta(g)^{1/2} \quad \text{and} \tag{2.4}$$

$$\dim_{R(K_1)-}(\mathscr{K}) = \sum_{g \in K_1 \backslash G/K_2} \dim(1_{eK_1} \cdot \mathscr{H}' \cdot 1_{K_1 g K_2})\, \Delta(g)^{-1/2} . \tag{2.5}$$

Also note that for every $g \in G$, we have

$$\begin{aligned}
\dim(1_{K_2 g^{-1} K_1} \cdot \mathscr{H}' \cdot 1_{eK_2}) &= [K_2 : K_2 \cap g^{-1} K_1 g] \; \dim(1_{g^{-1} K_1} \cdot \mathscr{H}' \cdot 1_{eK_2}) \\
&= [K_2 : K_2 \cap g^{-1} K_1 g] \; \dim(1_{eK_1} \cdot \mathscr{H}' \cdot 1_{gK_2}) \\
&= \frac{[K_2 : K_2 \cap g^{-1} K_1 g]}{[K_1 : K_1 \cap g K_2 g^{-1}]} \; \dim(1_{eK_1} \cdot \mathscr{H}' \cdot 1_{K_1 g K_2}) \\
&= [K_2 : K_1] \; \Delta(g)^{-1} \; \dim(1_{eK_1} \cdot \mathscr{H}' \cdot 1_{K_1 g K_2}) \; .
\end{aligned}$$

It follows that

$$\begin{aligned}
\dim_{-R(K_2)}(\mathscr{K}) &= [K_2 : K_1] \sum_{g \in K_1 \backslash G / K_2} \dim(1_{eK_1} \cdot \mathscr{H}' \cdot 1_{K_1 g K_2}) \, \Delta(g)^{-1/2} \\
&= [K_2 : K_1] \; \dim_{R(K_1)-}(\mathscr{K}) \; .
\end{aligned}$$

If $\mathscr{H}$ has finite rank, also $\mathscr{H}'$ has finite rank so that $\mathscr{H}' \cdot 1_{eK_2}$ and $1_{eK_1} \cdot \mathscr{H}'$ are finite dimensional Hilbert spaces. It then follows that $\mathscr{K}$ is a finite index bimodule.

Conversely, assume that $\mathscr{K}$ has finite index. For every $g \in G$, write

$$\begin{aligned}
\kappa(g) &:= \dim(1_{K_2 g^{-1} K_1} \cdot \mathscr{H}' \cdot 1_{eK_2}) \, \Delta(g)^{1/2} \\
&= [K_2 : K_1] \; \dim(1_{eK_1} \cdot \mathscr{H}' \cdot 1_{K_1 g K_2}) \, \Delta(g)^{-1/2} \; .
\end{aligned}$$

So,

$$\kappa(g)^2 = [K_2 : K_1] \; \dim(1_{K_2 g^{-1} K_1} \cdot \mathscr{H}' \cdot 1_{eK_2}) \; \dim(1_{eK_1} \cdot \mathscr{H}' \cdot 1_{K_1 g K_2}) \; .$$

Thus, whenever $\kappa(g) \neq 0$, we have that $\kappa(g) \geq [K_2 : K_1]^{1/2}$. Since

$$\dim_{-R(K_2)}(\mathscr{K}) = \sum_{g \in K_1 \backslash G / K_2} \kappa(g) \; ,$$

we conclude that there are only finitely many double cosets $g \in K_1 \backslash G / K_2$ for which $1_{K_2 g^{-1} K_1} \cdot \mathscr{H}' \cdot 1_{eK_2}$ is nonzero and for each of them, it is a finite dimensional Hilbert space. This implies that $\mathscr{H}' \cdot 1_{eK_2}$ is finite dimensional, so that $\mathscr{H}'$ has finite rank.

We have proved that $\mathscr{H} \mapsto \mathscr{K}$ is a fully faithful 2-functor from $\mathscr{C}_{1f}$ to the finite index bimodules over hyperfinite II_1 factors. Moreover, for given compact open subgroups $K_1, K_2 < G$, the ratio between $\dim_{R(K_1)-}(\mathscr{K})$ and $\dim_{-R(K_2)}(\mathscr{K})$ equals $[K_1 : K_2]$ for all finite rank K_1-K_2-$L^\infty(G)$-modules $\mathscr{H}$. Since the functor is fully faithful, this then also holds for all $R(K_1)$-$R(K_2)$-subbimodules of $\mathscr{K}$. It follows that the categorical dimension of $\mathscr{K}$ equals

$$[K_2 : K_1]^{1/2} \; \dim_{R(K_1)-}(\mathscr{K}) = [K_1 : K_2]^{1/2} \; \dim_{-R(K_2)}(\mathscr{K}) \; .$$

Since the functor is fully faithful, the categorical dimensions of $\mathscr{H} \in \mathscr{C}_{1,f}$ and $\mathscr{K} \in \mathrm{Bimod}_f$ coincide, so that

$$[K_2 : K_1]^{1/2} \dim_{R(K_1)-}(\mathscr{K}) = \dim_{\mathscr{C}_1}(\mathscr{H}) = [K_1 : K_2]^{1/2} \dim_{-R(K_2)}(\mathscr{K}) \; . \tag{2.6}$$

□

Corollary 2.4 *Let G be a totally disconnected group with compact open subgroups $K_\pm < G$ and assume that $\mathscr{H}$ is a finite rank G-$L^\infty(G/K_+)$-$L^\infty(G/K_-)$-module. Denote by $\mathscr{C} = (\mathscr{C}_{++}, \mathscr{C}_{+-}, \mathscr{C}_{-+}, \mathscr{C}_{--})$ the C*-2-category of G-$L^\infty(G/K_\pm)$-$L^\infty(G/K_\pm)$-modules (with 0-cells K_+ and K_-) generated by the alternating tensor products of $\mathscr{H}$ and its adjoint.*

Combining Proposition 2.2 and Theorem 2.3, we find an extremal hyperfinite subfactor $N \subset M$ whose standard invariant, viewed as the C-2-category of N-N, N-M, M-N and M-M-bimodules generated by the N-M-bimodule $L^2(M)$, is equivalent with $(\mathscr{C}, \mathscr{H})$ (cf. Remark 2.1).*

Proof A combination of Proposition 2.2 and Theorem 2.3 provides the finite index $R(K_+)$-$R(K_-)$-bimodule $\mathscr{K}$ associated with $\mathscr{H}$. Take nonzero projections $p_\pm \in R(K_\pm)$ such that writing $N = p_+ R(K_+) p_+$ and $M = p_- R(K_-) p_-$, we have that $\dim_{-M}(p_+ \cdot \mathscr{K} \cdot p_-) = 1$. We can then view $N \subset M$ in such a way that $L^2(M) \cong p_+ \cdot \mathscr{K} \cdot p_-$ as N-M-bimodules. The C*-2-category of N-N, N-M, M-N and M-M-bimodules generated by the N-M-bimodule $L^2(M)$ is by construction equivalent with the rigid C*-2-category of $R(K_\pm)$-$R(K_\pm)$-bimodules generated by $\mathscr{K}$. Since the 2-functor in Theorem 2.3 is fully faithful, this C*-2-category is equivalent with $\mathscr{C}$ and this equivalence maps the N-M-bimodule $L^2(M)$ to $\mathscr{H} \in \mathscr{C}_{+-}$. □

From Corollary 2.4, we get the following result.

Proposition 2.5 *Let $\mathscr{P}$ be the subfactor planar algebra of [5, 11] associated with a connected locally finite bipartite graph $\mathscr{G}$, with edge set $\mathscr{E}$ and source and target maps $s : \mathscr{E} \to V_+$, $t : \mathscr{E} \to V_-$, together with[2] a closed subgroup $G <$ Aut$(\mathscr{G})$ acting transitively on V_+ as well as on V_-. Fix vertices $v_\pm \in V_\pm$ and write $K_\pm =$ Stab $v_\pm$.*

There exists an extremal hyperfinite subfactor $N \subset M$ whose standard invariant is isomorphic with $\mathscr{P}$. We have $[M : N] = \delta^2$ where

$$\begin{aligned} \delta &= \sum_{w \in V_-} \#\{e \in \mathscr{E} \mid s(e) = v_+, t(e) = w\} \, [\mathrm{Stab}\, w : \mathrm{Stab}\, v_+]^{1/2} \\ &= \sum_{w \in V_+} \#\{e \in \mathscr{E} \mid s(e) = w, t(e) = v_-\} \, [\mathrm{Stab}\, w : \mathrm{Stab}\, v_-]^{1/2} \; . \end{aligned}$$

[2]Note that in [5], also a weight function $\mu : V_+ \sqcup V_- \to \mathbb{R}_0^+$ scaled by the action of G is part of the construction. But only when we take μ to be a multiple of the function $v \mapsto [\mathrm{Stab}\, v : \mathrm{Stab}\, v_+]^{1/2}$, we actually obtain a subfactor planar algebra, contrary to what is claimed in [5, Proposition 4.1].

Moreover, $\mathscr{P}$ can be described as the rigid C-2-category $\mathscr{C}_{3,f}(G, K_\pm, K_\pm)$ of all finite rank G-$L^\infty(G/K_\pm)$-$L^\infty(G/K_\pm)$-modules together with the generating object $\ell^2(\mathscr{E}) \in \mathscr{C}_{3,f}(G, K_+, K_-)$ (cf. Remark 2.1).*

Proof We are given $G \curvearrowright \mathscr{E}$ and $G \curvearrowright V_+$, $G \curvearrowright V_-$ such that the source and target maps s, t are G-equivariant and such that G acts transitively on V_+ and on V_-. Put $K_\pm = \operatorname{Stab} v_\pm$ and note that $K_\pm < G$ are compact open subgroups. We identify $G/K_\pm = V_\pm$ via the map $gK_\pm \mapsto g \cdot v_\pm$. In this way, $\mathscr{H} := \ell^2(\mathscr{E})$ naturally becomes a finite rank G-$L^\infty(G/K_+)$-$L^\infty(G/K_-)$-module. Denote by $\mathscr{C}$ the C*-2-category of G-$L^\infty(G/K_\pm)$-$L^\infty(G/K_\pm)$-modules generated by the alternating tensor products of $\mathscr{H}$ and its adjoint.

In the 2-category $\mathscr{C}_3$, the n-fold tensor product $\mathscr{H} \otimes \overline{\mathscr{H}} \otimes \cdots$ equals $\ell^2(\mathscr{E}_{+,n})$, where $\mathscr{E}_{+,n}$ is the set of paths in the graph $\mathscr{G}$ starting at an even vertex and having length n. Similarly, the n-fold tensor product $\overline{\mathscr{H}} \otimes \mathscr{H} \otimes \cdots$ equals $\ell^2(\mathscr{E}_{-,n})$, where $\mathscr{E}_{-,n}$ is the set of paths of length n starting at an odd vertex. So by construction, under the equivalence of Remark 2.1, $\mathscr{C}$ together with its generator $\mathscr{H} \in \mathscr{C}_{+-}$ corresponds exactly to the planar algebra $\mathscr{P}$ constructed in [5, 11].

By Corollary 2.4, we get that $(\mathscr{C}, \mathscr{H})$ is the standard invariant of an extremal hyperfinite subfactor $N \subset M$. In particular, $[M : N] = \delta^2$ with $\delta = \dim_{\mathscr{C}_3}(\mathscr{H})$. Combining (2.6) with (2.4), and using that

$$\Delta(g)^{-1/2} = [gK_+g^{-1} : K_+]^{1/2} = [\operatorname{Stab}(g \cdot v_+) : K_+]^{1/2} ,$$

we get that

$$\begin{aligned}\delta &= [K_+ : K_-]^{1/2} \sum_{g \in G/K_+} \dim(1_{gK_+} \cdot \mathscr{H} \cdot 1_{eK_-})\, \Delta(g)^{-1/2} \\ &= \sum_{g \in G/K_+} \#\{e \in \mathscr{E} \mid s(e) = g \cdot v_+, t(e) = v_-\}\, [\operatorname{Stab}(g \cdot v_+) : K_+]^{1/2}\, [K_+ : K_-]^{1/2} \\ &= \sum_{w \in V_+} \#\{e \in \mathscr{E} \mid s(e) = w, t(e) = v_-\}\, [\operatorname{Stab} w : \operatorname{Stab} v_-]^{1/2} .\end{aligned}$$

Combining (2.6) with (2.5), we similarly get that

$$\delta = \sum_{w \in V_-} \#\{e \in \mathscr{E} \mid s(e) = v_+, t(e) = w\}\, [\operatorname{Stab} w : \operatorname{Stab} v_+]^{1/2} .$$

To conclude the proof of the proposition, it remains to show that $\mathscr{C}$ is equal to the C*-2-category of all finite rank G-$L^\infty(G/K_\pm)$-$L^\infty(G/K_\pm)$-modules. For the G-$L^\infty(G/K_+)$-$L^\infty(G/K_-)$-modules, this amounts to proving that all irreducible representations of $K_+ \cap K_-$ appear in

$$\ell^2(\text{paths starting at } v_+ \text{ and ending at } v_-) .$$

Since the graph is connected, the action of $K_+ \cap K_-$ on this set of paths is faithful and the result follows. The other cases are proved in the same way. □

Remark 2.6 Note that the subfactors $N \subset M$ in Proposition 2.5 are *irreducible* precisely when G acts transitively on the set of edges and there are no multiple edges. This means that the totally disconnected group G is *generated* by the compact open subgroups $K_\pm < G$ and that we can identify $\mathscr{E} = G/(K_+ \cap K_-)$, $V_\pm = G/K_\pm$ with the natural source and target maps $G/(K_+ \cap K_-) \to G/K_\pm$. The irreducible subfactor $N \subset M$ then has integer index given by $[M : N] = [K_+ : K_+ \cap K_-]\,[K_- : K_+ \cap K_-]$.

We finally note that the rigid C*-tensor categories $\mathscr{C}_{1,f}(K < G)$ and $\mathscr{C}_{3,f}(K < G)$ also arise in a different way as categories of bimodules over a II_1 factor in the case where $K < G$ is the *Schlichting completion* of a *Hecke pair* $\Lambda < \Gamma$, cf. [7, Section 4].

Recall that a Hecke pair consists of a countable group Γ together with a subgroup $\Lambda < \Gamma$ that is almost normal, meaning that $g\Lambda g^{-1} \cap \Lambda$ has finite index in Λ for all $g \in \Gamma$. The left translation action of Γ on Γ/Λ gives a homomorphism π of Γ to the group of permutations of Γ/Λ. The closure G of $\pi(\Gamma)$ for the topology of pointwise convergence is a totally disconnected group and the stabilizer K of the point $e\Lambda \in \Gamma/\Lambda$ is a compact open subgroup of G with the property that $\Lambda = \pi^{-1}(K)$. One calls (G, K) the Schlichting completion of the Hecke pair (Γ, Λ). Note that there is a natural identification of G/K and Γ/Λ.

Proposition 2.7 *Let $\Lambda < \Gamma$ be a Hecke pair with Schlichting completion $K < G$. Choose an action $\Gamma \curvearrowright^{\alpha} P$ of Γ by outer automorphisms of a II_1 factor P. Define $N = P \rtimes \Lambda$ and $M = P \rtimes \Gamma$. Note that $N \subset M$ is an irreducible, quasi-regular inclusion of II_1 factors. Denote by $\mathscr{C}$ the tensor category of finite index N-N-bimodules generated by the finite index N-subbimodules of $L^2(M)$.*

Then, $\mathscr{C}$ and the earlier defined $\mathscr{C}_{1,f}(K < G)$ and $\mathscr{C}_{3,f}(K < G)$ are naturally equivalent rigid C-tensor categories.*

Proof Define

- $\mathscr{C}_4$: the category of Λ-Λ-$\ell^\infty(\Gamma)$-modules, i.e. Hilbert spaces $\mathscr{H}$ equipped with two commuting unitary representations of Λ and a representation of $\ell^\infty(\Gamma)$ that are covariant with respect to the left and right translation actions $\Lambda \curvearrowright \Gamma$;
- $\mathscr{C}_5$: the category of Λ-$\ell^\infty(\Gamma/\Lambda)$-modules, i.e. Hilbert spaces equipped with a unitary representation of Λ and a representation of $\ell^\infty(\Gamma/\Lambda)$ that are covariant with respect to the left translation action $\Lambda \curvearrowright \Gamma/\Lambda$;

with morphisms again given by bounded operators that intertwine the given structure.

To define the tensor product of two objects in $\mathscr{C}_4$, it is useful to view $\mathscr{H} \in \mathscr{C}_4$ as a family of Hilbert spaces $(\mathscr{H}_g)_{g\in\Gamma}$ together with unitary operators $\lambda(k) : \mathscr{H}_g \to \mathscr{H}_{kg}$ and $\rho(k) : \mathscr{H}_g \to \mathscr{H}_{gk^{-1}}$ for all $k \in \Lambda$, satisfying the obvious relations. The tensor

product of two Λ-Λ-$\ell^\infty(\Gamma)$-modules $\mathscr{H}$ and $\mathscr{K}$ is then defined as

$$(\mathscr{H} \otimes_\Lambda \mathscr{K})_g = \Big\{ (\xi_h)_{h\in\Gamma} \;\Big|\; \xi_h \in \mathscr{H}_h \otimes \mathscr{K}_{h^{-1}g} \,, \quad \xi_{hk^{-1}} = (\rho_{\mathscr{H}}(k) \otimes \lambda_{\mathscr{K}}(k))(\xi_h) \text{ for all } h \in \Gamma, k \in \Lambda \,, \quad \sum_{h\in\Gamma/\Lambda} \|\xi_h\|^2 < \infty \Big\}$$

with $\lambda(k) : (\mathscr{H} \otimes_\Lambda \mathscr{K})_g \to (\mathscr{H} \otimes_\Lambda \mathscr{K})_{kg}$ given by $(\lambda(k)\xi)_h = (\lambda_{\mathscr{H}}(k) \otimes 1)\xi_{k^{-1}h}$ and $\rho(k) : (\mathscr{H} \otimes_\Lambda \mathscr{K})_g \to (\mathscr{H} \otimes_\Lambda \mathscr{K})_{gk^{-1}}$ given by $(\rho(k)\xi)_h = (1 \otimes \rho_{\mathscr{K}}(k))\xi(h)$ for all $k \in \Lambda$, $h \in \Gamma$. Of course, choosing a section $i : \Gamma/\Lambda \to \Gamma$, we have

$$(\mathscr{H} \otimes_\Lambda \mathscr{K})_g \cong \bigoplus_{h\in\Gamma/\Lambda} (\mathscr{H}_{i(h)} \otimes \mathscr{K}_{i(h)^{-1}g}) \,,$$

but this isomorphism depends on the choice of the section.

As in Proposition 2.2, $\mathscr{C}_4$ and $\mathscr{C}_5$ are equivalent C*-categories, where the equivalence and its inverse are defined as follows.

- $\mathscr{C}_4 \to \mathscr{C}_5 : \mathscr{H} \mapsto \mathscr{K}$, with

  $$\mathscr{K}_{g\Lambda} = \big\{ (\xi_h)_{h\in g\Lambda} \;\big|\; \xi_h \in \mathscr{H}_h \,,\; \xi_{hk^{-1}} = \rho(k)\xi_h \text{ for all } h \in g\Lambda, k \in \Lambda \big\}$$

  and with the natural Λ-$\ell^\infty(\Gamma/\Lambda)$-module structure. Note that $\mathscr{K}_{g\Lambda} \cong \mathscr{H}_g$, but again, this isomorphism depends on a choice of section $\Gamma/\Lambda \to \Gamma$.
- $\mathscr{C}_5 \to \mathscr{C}_4 : \mathscr{K} \mapsto \mathscr{H}$, with $\mathscr{H}_g = \mathscr{K}_{g\Lambda}$ and the obvious Λ-Λ-$\ell^\infty(\Gamma)$-module structure.

We say that an object $\mathscr{H} \in \mathscr{C}_5$ has finite rank if $\mathscr{H}$ is a finite dimensional Hilbert space. This is equivalent to requiring that all Hilbert spaces $\mathscr{H}_{g\Lambda}$ are finite dimensional and that there are only finitely many double cosets $\Lambda g \Lambda$ for which $\mathscr{H}_{g\Lambda}$ is nonzero. Similarly, we say that an object $\mathscr{H} \in \mathscr{C}_4$ has finite rank if all Hilbert spaces $\mathscr{H}_g$ are finite dimensional and if there are only finitely many double cosets $\Lambda g \Lambda$ for which $\mathscr{H}_g$ is nonzero. Note here that an algebraic variant of the category of finite rank objects in $\mathscr{C}_4$ was already introduced in [29].

In this way, we have defined the rigid C*-tensor category $\mathscr{C}_{4,f}(\Lambda < \Gamma)$ consisting of the finite rank objects in $\mathscr{C}_4$. Note that, in a different context, this rigid C*-tensor category $\mathscr{C}_{4,f}(\Lambda < \Gamma)$ already appeared in [7, Section 4].

Denote by $\pi : \Gamma \to G$ the canonical homomorphism. Identifying G/K and Γ/Λ and using the homomorphism $\pi : \Lambda \to K$, every K-$L^\infty(G/K)$-module $\mathscr{H}$ also is a Λ-$\ell^\infty(\Gamma/\Lambda)$-module. This defines a functor $\mathscr{C}_2(K < G) \to \mathscr{C}_5(\Lambda < \Gamma)$ that is fully faithful because $\pi(\Lambda)$ is dense in K. Note however that this fully faithful functor need not be an equivalence of categories: an object $\mathscr{H} \in \mathscr{C}_5(\Lambda < \Gamma)$ is isomorphic with an object in the range of this functor if and only if the representation of Λ on

$\mathscr{H}$ is of the form $k \mapsto \lambda(\pi(k))$ for a (necessarily unique) continuous representation λ of K on $\mathscr{H}$.

Composing with the equivalence of categories in Proposition 2.2, we have found the fully faithful C*-tensor functor $\Theta : \mathscr{C}_3(K < G) \to \mathscr{C}_4(\Lambda < \Gamma)$, sending finite rank objects to finite rank objects. By construction, Θ maps the G-$L^\infty(G/K)$-$L^\infty(G/K)$-module $L^2(G/K) \otimes L^2(G/K)$ (with G-action given by $(\lambda_g \otimes \lambda_g)_{g\in G}$ and obvious left and right $L^\infty(G/K)$-action) to the Λ-Λ-$\ell^\infty(\Gamma)$-module $\ell^2(\Gamma)$.

Next, given the outer action $\Gamma \curvearrowright^\alpha P$, we write $N = P \rtimes \Lambda$ and $M = P \rtimes \Gamma$. Consider the category Bimod(N) of Hilbert N-N-bimodules. We define the natural fully faithful C*-tensor functor $\mathscr{C}_4(\Lambda < \Gamma) \to \mathrm{Bimod}(N) : \mathscr{H} \mapsto \mathscr{K}$ where $\mathscr{K} = L^2(P) \otimes \mathscr{H}$ and where the N-N-bimodule structure on $\mathscr{K}$ is given by

$$(au_k) \cdot (b \otimes \xi) \cdot (du_r) = a\alpha_k(b)\alpha_{kh}(d) \otimes \lambda(k)\rho(r^{-1})\xi$$

for all $a, b, d \in P$, $k, r \in \Lambda$, $h \in \Gamma$ and $\xi \in \mathscr{H}_h$. By construction, this functor maps the Λ-Λ-$\ell^\infty(\Gamma)$-module $\ell^2(\Gamma)$ to the N-N-bimodule $L^2(M)$.

Denoting by $\mathscr{C}$ the tensor category of finite index N-N-bimodules generated by the finite index N-subbimodules of $L^2(M)$, it follows that $\mathscr{C}$ is naturally monoidally equivalent to the tensor subcategory $\mathscr{C}_0$ of $\mathscr{C}_{3,f}(K < G)$ generated by the finite rank subobjects of $L^2(G/K) \otimes L^2(G/K)$. So, it remains to prove that $\mathscr{C}_0 = \mathscr{C}_{3,f}(K < G)$. Taking the n-th tensor power of $L^2(G/K) \otimes L^2(G/K)$ and applying the equivalence between the categories $\mathscr{C}_{3,f}(K < G)$ and $\mathscr{C}_{2,f}(K < G)$, it suffices to show that every irreducible K-$L^\infty(G/K)$-module appears in one of the K-$L^\infty(G/K)$-modules $L^2(G/K) \otimes \cdots \otimes L^2(G/K)$ with diagonal G-action and action of $L^\infty(G/K)$ on the last tensor factor. Reducing with the projections 1_{gK}, this amounts to proving that for every $g \in G$, every irreducible representation of the compact group $K \cap gKg^{-1}$ appears in a tensor power of $L^2(G/K)$. Because $K < G$ is a Schlichting completion, we have that $\bigcap_{h\in G} hKh^{-1} = \{e\}$ so that the desired conclusion follows. $\square$

3 The Tube Algebra of $\mathscr{C}(K < G)$

Recall from [19] the following construction of the *tube $*$-algebra* of a rigid C*-tensor category $\mathscr{C}$ (see also [9, Section 3] where the terminology *annular algebra* is used, and see as well [25, Section 3.3]). Whenever I is a full[3] family of objects in $\mathscr{C}$, one defines as follows the $*$-algebra $\mathscr{A}$ with underlying vector space

$$\mathscr{A} = \bigoplus_{i,j\in I, \alpha\in \mathrm{Irr}(\mathscr{C})} (i\alpha, \alpha j) .$$

[3]Fullness means that every irreducible $i \in \mathrm{Irr}(\mathscr{C})$ appears as a subobject of one of the $j \in I$.

Here and in what follows, we denote the tensor product in $\mathscr{C}$ by concatenation and we denote by (β, γ) the space of morphisms from γ to β. By definition, all (β, γ) are finite dimensional Banach spaces. Using the categorical traces Tr_β and Tr_γ on (β, β), resp. (γ, γ), we turn (β, γ) into a Hilbert space with scalar product

$$\langle V, W \rangle = \mathrm{Tr}_\beta(VW^*) = \mathrm{Tr}_\gamma(W^* V) \; .$$

For every $\beta \in \mathscr{C}$, the categorical trace Tr_β is defined by using a standard solution for the conjugate equations for β, i.e. morphisms $s_\beta \in (\beta\overline{\beta}, \varepsilon)$ and $t_\beta \in (\overline{\beta}\beta, \varepsilon)$ satisfying

$$(s_\beta^* \otimes 1)(1 \otimes t_\beta) = 1 \;\; , \;\; (1 \otimes s_\beta^*)(t_\beta \otimes 1) = 1 \;\; , \;\; t_\beta^*(1 \otimes V)t_\beta = s_\beta^*(V \otimes 1)s_\beta$$

for all $V \in (\beta, \beta)$. Then, $\mathrm{Tr}_\beta(V) = t_\beta^*(1 \otimes V)t_\beta = s_\beta^*(V \otimes 1)s_\beta$ and $d(\beta) = \mathrm{Tr}_\beta(1)$ is the categorical dimension of β.

We will also make use of the partial traces

$$\mathrm{Tr}_\beta \otimes \mathrm{id} : (\beta\alpha, \beta\gamma) \to (\alpha, \gamma) : (\mathrm{Tr}_\beta \otimes \mathrm{id})(V) = (t_\beta^* \otimes 1)(1 \otimes V)(t_\beta \otimes 1) \; .$$

Whenever $\mathscr{K}$ is a Hilbert space, we denote by $\mathrm{onb}(\mathscr{K})$ any choice of orthonormal basis in $\mathscr{K}$. The product in $\mathscr{A}$ is then defined as follows: for $V \in (i\alpha, \alpha j)$ and $W \in (j'\beta, \beta, k)$, the product $V \cdot W$ equals 0 when $j \neq j'$ and when $j = j'$, it is equal to

$$V \cdot W = \sum_{\gamma \in \mathrm{Irr}(\mathscr{C})} \; \sum_{U \in \mathrm{onb}(\alpha\beta, \gamma)} d(\gamma)\, (1 \otimes U^*)(V \otimes 1)(1 \otimes W)(U \otimes 1) \; .$$

The $*$-operation on $\mathscr{A}$ is denoted by $V \mapsto V^\#$ and defined by

$$V^\# = (t_\alpha^* \otimes 1)(1 \otimes V^* \otimes 1)(1 \otimes s_\alpha)$$

for all $V \in (i\alpha, \alpha j)$.

The $*$-algebra $\mathscr{A}$ has a natural positive faithful trace Tr and for $V \in (i\alpha, \alpha j)$, we have that $\mathrm{Tr}(V) = 0$ when $i \neq j$ or $\alpha \neq \varepsilon$, while $\mathrm{Tr}(V) = \mathrm{Tr}_i(V)$ when $i = j$ and $\alpha = \varepsilon$, so that $V \in (i, i)$.

Up to strong Morita equivalence, the tube $*$-algebra $\mathscr{A}$ does not depend on the choice of the full family I of objects in $\mathscr{C}$, see [18, Theorem 3.2] and [25, Section 7.2]. Also note that for an arbitrary object $\alpha \in \mathscr{C}$ and $i, j \in I$, we can associate with $V \in (i\alpha, \alpha j)$ the element in $\mathscr{A}$ given by

$$\sum_{\gamma \in \mathrm{Irr}(\mathscr{C})} \; \sum_{U \in \mathrm{onb}(\alpha, \gamma)} d(\gamma)\, (1 \otimes U^*)V(U \otimes 1) \; .$$

Although this map $(i\alpha, \alpha j) \to \mathscr{A}$ is not injective, we will view an element in $V \in (i\alpha, \alpha j)$ as an element of $\mathscr{A}$ in this way.

Formally allowing for infinite direct sums in $\mathscr{C}$, one defines the C*-tensor category of ind-objects in $\mathscr{C}$. Later in this section, we will only consider the rigid C*-tensor category $\mathscr{C}$ of finite rank G-$L^\infty(G/K)$-$L^\infty(G/K)$-modules for a given totally disconnected group G with compact open subgroup $K < G$. In that case, the ind-category precisely[4] is the C*-tensor category of all G-$L^\infty(G/K)$-$L^\infty(G/K)$-modules. Whenever $\mathscr{K}_1, \mathscr{K}_2$ are ind-objects, we denote by $(\mathscr{K}_1, \mathscr{K}_2)$ the vector space of *finitely supported* morphisms, where a morphism $V : \mathscr{K}_2 \to \mathscr{K}_1$ is said to be finitely supported if there exist projections p_i of $\mathscr{K}_i$ onto a finite dimensional subobject (i.e. an object in $\mathscr{C}$) such that $V = p_1 V = V p_2$.

We say that an ind-object $\mathscr{H}_0$ in $\mathscr{C}$ is full if every irreducible object $i \in \mathrm{Irr}(\mathscr{C})$ is isomorphic with a subobject of $\mathscr{H}_0$. We define the tube $*$-algebra of $\mathscr{C}$ with respect to a full ind-object $\mathscr{H}_0$ as the vector space

$$\mathscr{A} = \bigoplus_{\alpha \in \mathrm{Irr}(\mathscr{C})} (\mathscr{H}_0 \alpha, \alpha \mathscr{H}_0)$$

on which the $*$-algebra structure is defined in the same way as above. Note that $(\mathscr{H}_0, \mathscr{H}_0)$ naturally is a $*$-subalgebra of $\mathscr{A}$, given by taking $\alpha = \varepsilon$ in the above description of $\mathscr{A}$. In particular, every projection of p of $\mathscr{H}_0$ on a finite dimensional subobject of $\mathscr{H}_0$ can be viewed as a projection $p \in \mathscr{A}$. These projections serve as local units: for every finite subset $\mathscr{F} \subset \mathscr{A}$, there exists such a projection p satisfying $p \cdot V = V \cdot p$ for all $V \in \mathscr{F}$.

Whenever p_ε is the projection of $\mathscr{H}_0$ onto a copy of the trivial object ε, we identify $p_\varepsilon \cdot \mathscr{A} \cdot p_\varepsilon$ with the fusion $*$-algebra $\mathbb{C}[\mathscr{C}]$ of $\mathscr{C}$, i.e. the $*$-algebra with vector space basis $\mathrm{Irr}(\mathscr{C})$, product given by the fusion rules and $*$-operation given by the adjoint object.

To every full family I of objects in $\mathscr{C}$, we can associate the full ind-object $\mathscr{H}_0$ by taking the direct sum of all $i \in I$. The tube $*$-algebra of $\mathscr{C}$ associated with I is then naturally a $*$-subalgebra of the tube $*$-algebra of $\mathscr{C}$ associated with $\mathscr{H}_0$. If every irreducible object of $\mathscr{C}$ appears with finite multiplicity in $\mathscr{H}_0$, then this inclusion is an equality and both tube $*$-algebras are naturally isomorphic.

For the rest of this section, we fix a totally disconnected group G and a compact open subgroup $K < G$. We denote by $\mathscr{C}$ the rigid C*-tensor category of all finite rank G-$L^\infty(G/K)$-$L^\infty(G/K)$-modules, which we denoted as $\mathscr{C}_{3,f}(K < G)$ in Sect. 2. We determine the tube $*$-algebra $\mathscr{A}$ of $\mathscr{C}$ with respect to the following full ind-object.

[4]Using Proposition 2.2, every G-$L^\infty(G/K)$-$L^\infty(G/K)$-module is a direct sum of finite rank modules because every K-$L^\infty(G/K)$-module is a direct sum of finite dimensional modules, which follows because every unitary representation of a compact group is a direct sum of finite dimensional representations.

$$\begin{aligned}
&\mathscr{H}_0 = L^2(G \times G/K) \quad \text{with} \\
&(F \cdot \xi)(g, hK) = F(gK)\, \xi(g, hK) \;\; , \\
&(\xi \cdot F)(g, hK) = \xi(g, hK)\, F(ghK) \;\; \text{and} \\
&(\pi(x)\xi)(g, hK) = \xi(x^{-1}g, hK)
\end{aligned} \tag{3.1}$$

for all $\xi \in L^2(G \times G/K)$, $F \in L^\infty(G/K)$, $x, g \in G$, $hK \in G/K$. Note that every irreducible object of $\mathscr{C}$ appears with finite multiplicity in $\mathscr{H}_0$.

We denote by $(\mathrm{Ad}\, g)_{g \in G}$ the action of G on G by conjugation: $(\mathrm{Ad}\, g)(h) = ghg^{-1}$. In the rest of this paper, we will make use of the associated full and reduced C*-algebras

$$C_0(G) \rtimes^f_{\mathrm{Ad}} G \quad \text{and} \quad C_0(G) \rtimes^r_{\mathrm{Ad}} G \;\; ,$$

as well as the von Neumann algebra $L^\infty(G) \rtimes_{\mathrm{Ad}} G$. We fix the left Haar measure λ on G such that $\lambda(K) = 1$. We equip $L^\infty(G) \rtimes_{\mathrm{Ad}} G$ with the canonical normal semifinite faithful trace Tr given by

$$\mathrm{Tr}(F\lambda_f) = f(e) \int_G F(g) \Delta(g)^{-1/2}\, dg \; . \tag{3.2}$$

Note that the modular function Δ is affiliated with the center of $L^\infty(G) \rtimes_{\mathrm{Ad}} G$, so that $L^\infty(G) \rtimes_{\mathrm{Ad}} G$ need not be a factor. Also note that the measure used in (3.2) is half way between the left and the right Haar measure of G.

We consider the dense $*$-algebra $\mathrm{Pol}(L^\infty(G) \rtimes_{\mathrm{Ad}} G)$ defined as

$$\begin{aligned}
\mathrm{Pol}(L^\infty(G) \rtimes_{\mathrm{Ad}} G) = \mathrm{span}\{1_{\mathscr{U}}\, u_x\, p_L \mid \; &\mathscr{U} \subset G \;\text{ compact open subset}\;,\; x \in G\;, \\
&L < G \;\text{ compact open subgroup}\}
\end{aligned} \tag{3.3}$$

and where $p_L \in L(G)$ denotes the projection onto the L-invariant vectors, i.e.

$$p_L = \lambda(L)^{-1} \int_L u_k\, dk \; .$$

Note that $\mathrm{Pol}(L^\infty(G) \rtimes_{\mathrm{Ad}} G)$ equals the linear span of all $F\lambda_f$ where F and f are continuous, compactly supported, locally constant functions on G.

We now identify the tube $*$-algebra of $\mathscr{C}$ with $\mathrm{Pol}(L^\infty(G) \rtimes_{\mathrm{Ad}} G)$. For every $x \in G$ and every irreducible representation $\pi : K \cap xKx^{-1} \to \mathscr{U}(\mathscr{K})$, we denote by $\mathscr{H}(\pi, x) \in \mathrm{Irr}(\mathscr{C})$ the irreducible G-$L^\infty(G/K)$-$L^\infty(G/K)$-module such that π is isomorphic with the representation of $K \cap xKx^{-1}$ on $1_{xK} \cdot \mathscr{H}(\pi, x) \cdot 1_{eK}$. Note that this gives us the identification

$$\mathrm{Irr}(\mathscr{C}) = \{(\pi, x) \mid x \in K\backslash G/K \;,\; \pi \in \mathrm{Irr}(K \cap xKx^{-1})\} \; . \tag{3.4}$$

We denote by χ_π the character of π, i.e. the locally constant function with support $K \cap xKx^{-1}$ and $\chi_\pi(k) = \mathrm{Tr}(\pi(k))$ for all $k \in K \cap xKx^{-1}$.

Theorem 3.1 *The G-$L^\infty(G/K)$-$L^\infty(G/K)$-module $\mathscr{H}_0$ introduced in (3.1) is full. There is a natural $*$-anti-isomorphism Θ of the associated tube $*$-algebra $\mathscr{A}$ onto the $*$-algebra $\mathrm{Pol}(L^\infty(G) \rtimes_{\mathrm{Ad}} G)$. The $*$-anti-isomorphism Θ is trace preserving.*

Denoting by p_ε the projection in $\mathscr{A}$ that corresponds to the unique copy of the trivial object ε in $\mathscr{H}_0$ and identifying $p_\varepsilon \cdot \mathscr{A} \cdot p_\varepsilon$ with the fusion $*$-algebra of $\mathscr{C}$, we have that $\Theta(p_\varepsilon) = 1_K p_K$ and that the restriction of Θ to $\mathbb{C}[\mathscr{C}]$ is given by

$$d(\pi,x)^{-1}\Theta(\pi,x) = p_K \ \dim(\pi)^{-1}\chi_\pi \, u_x \, p_K \ , \tag{3.5}$$

where $d(\pi,x)$ denotes the categorical dimension of $(\pi,x) \in \mathrm{Irr}(\mathscr{C})$ and $\dim(\pi)$ denotes the ordinary dimension of the representation π.

Proof To see that $\mathscr{H}_0$ is full, it suffices to observe that for every $h \in G$, the unitary representation of $K \cap hKh^{-1}$ on $1_{eK} \cdot \mathscr{H}_0 \cdot 1_{hK}$ contains the regular representation of $K \cap hKh^{-1}$.

Assume that $\Psi : C_0(G) \rtimes^f_{\mathrm{Ad}} G \to B(\mathscr{K})$ is any nondegenerate $*$-representation. As follows, we associate with Ψ a unitary half braiding[5] on ind-$\mathscr{C}$. Whenever $\mathscr{H}$ is a G-$L^\infty(G/K)$-$L^\infty(G/K)$-module, we consider a new G-$L^\infty(G/K)$-$L^\infty(G/K)$-module with underlying Hilbert space $\mathscr{K} \otimes \mathscr{H}$ and structure maps

$$\begin{aligned}
\pi_{\mathscr{K}\otimes\mathscr{H}}(g) &= \Psi(g) \otimes \pi_{\mathscr{H}}(g) \ , \\
\lambda_{\mathscr{K}\otimes\mathscr{H}}(F) &= (\Psi \otimes \lambda_{\mathscr{H}})\Delta(F) \ , \\
\rho_{\mathscr{K}\otimes\mathscr{H}}(F) &= 1 \otimes \rho_{\mathscr{H}}(F) \ ,
\end{aligned}$$

for all $g \in G$, $F \in L^\infty(G/K)$, with $\Delta(F)(g, hK) = F(ghK)$.

We similarly turn $\mathscr{H} \otimes \mathscr{K}$ into a G-$L^\infty(G/K)$-$L^\infty(G/K)$-module with structure maps

$$\begin{aligned}
\pi_{\mathscr{H}\otimes\mathscr{K}}(g) &= \pi_{\mathscr{H}}(g) \otimes \Psi(g) \ , \\
\lambda_{\mathscr{H}\otimes\mathscr{K}}(F) &= \lambda_{\mathscr{H}}(F) \otimes 1 \ , \\
\rho_{\mathscr{H}\otimes\mathscr{K}}(F) &= (\rho_{\mathscr{H}} \otimes \Psi)\widetilde{\Delta}(F) \ ,
\end{aligned}$$

where $\widetilde{\Delta}(F)(gK, h) = F(h^{-1}gK)$.

Defining the unitary $U \in M(C_0(G) \otimes K(\mathscr{H}))$ given by $U(x) = \pi_{\mathscr{H}}(x)$ for all $x \in G$ and denoting by $\Sigma : \mathscr{K} \otimes \mathscr{H} \to \mathscr{H} \otimes \mathscr{K}$ the flip map, one checks that

[5] Formally, a unitary half braiding is an object in the Drinfeld center of ind-$\mathscr{C}$. More concretely, a unitary half braiding consists of an underlying ind-object $\mathscr{K}_1$ together with natural unitary isomorphisms $\mathscr{H}\mathscr{K}_1 \to \mathscr{K}_1\mathscr{H}$ for all objects $\mathscr{H}$. We refer to [17, Section 2.1] for further details.

the unitary $\Sigma(\Psi \otimes \mathrm{id})(U)$ is an isomorphism between the G-$L^\infty(G/K)$-$L^\infty(G/K)$-modules $\mathscr{K} \otimes \mathscr{H}$ and $\mathscr{H} \otimes \mathscr{K}$. So, defining

$$\mathscr{K}_1 := \mathscr{K} \otimes L^2(G/K) \cong L^2(G/K) \otimes \mathscr{K} \; ,$$

we have found the G-$L^\infty(G/K)$-$L^\infty(G/K)$-module $\mathscr{K}_1$ with the property that for every G-$L^\infty(G/K)$-$L^\infty(G/K)$-module $\mathscr{H}$, there is a natural unitary isomorphism

$$\sigma_{\mathscr{H}} : \mathscr{H}\mathscr{K}_1 \to \mathscr{K}_1\mathscr{H} \; .$$

Here and in what follows, we denote by concatenation the tensor product in the category of G-$L^\infty(G/K)$-$L^\infty(G/K)$-modules. So, σ is a unitary half braiding for ind-$\mathscr{C}$.

Using the ind-object $\mathscr{H}_0$ defined in (3.1) and recalling that $\mathscr{K}_1\overline{\mathscr{H}_0} = \mathscr{K} \otimes \overline{\mathscr{H}_0}$ as Hilbert spaces, we define the Hilbert space

$$\mathscr{K}_2 = (\mathscr{K} \otimes \overline{\mathscr{H}_0}, \varepsilon)$$

and we consider the tube $*$-algebra $\mathscr{A}$ associated with $\mathscr{H}_0$. Using standard solutions for the conjugate equations, there is a natural linear bijection

$$V \in (\mathscr{H}_0\mathscr{H}, \mathscr{H}\mathscr{H}_0) \mapsto \widetilde{V} \in (\mathscr{H}\overline{\mathscr{H}_0}, \overline{\mathscr{H}_0}\mathscr{H})$$

between finitely supported morphisms.

By [25, Proposition 3.14], using the partial categorical trace $\mathrm{Tr}_{\mathscr{H}} \otimes \mathrm{id} \otimes \mathrm{id}$, the unitary half braiding σ gives rise to a nondegenerate $*$-anti-homomorphism $\Theta : \mathscr{A} \to B(\mathscr{K}_2)$ given by

$$\Theta(V)\xi = (\mathrm{Tr}_{\mathscr{H}} \otimes \mathrm{id} \otimes \mathrm{id})\big((\sigma_{\mathscr{H}}^* \otimes 1)(1 \otimes \widetilde{V})(\xi \otimes 1)\big) \tag{3.6}$$

for all $\mathscr{H} \in \mathscr{C}$, $\xi \in \mathscr{K}_2$ and all finitely supported $V \in (\mathscr{H}_0\mathscr{H}, \mathscr{H}\mathscr{H}_0)$.

We now compute the expression in (3.6) more concretely. Whenever $h \in G$ and $K_0 < K$ is an open subgroup such that $hK_0h^{-1} \subset K$, we define the finite rank G-$L^\infty(G/K)$-$L^\infty(G/K)$-module $L^2(G/K_0)_h$ with underlying Hilbert space $L^2(G/K_0)$ and structure maps

$$(x \cdot \xi)(gK_0) = \xi(x^{-1}gK_0) \quad , \quad (F_1 \cdot \xi \cdot F_2)(gK_0) = F_1(gK)\, \xi(gK_0)\, F_2(gh^{-1}K) \; .$$

Note that there is a natural isomorphism $\overline{L^2(G/K_0)_h} \cong L^2(G/K_0)_{h^{-1}}$. Letting K_0 tend to $\{e\}$, the direct limit of $L^2(G/K_0)_{h^{-1}}$ becomes $L^2(G)_{h^{-1}}$. Since $\mathscr{H}_0 = \bigoplus_{h \in G/K} L^2(G)_{h^{-1}}$, we identify

$$\overline{\mathscr{H}_0} = \bigoplus_{h \in G/K} L^2(G)_h$$

and we view $L^2(G/K_0)_h \subset \overline{\mathscr{H}_0}$ whenever $h \in G$ and $K_0 < K \cap h^{-1}Kh$ is an open subgroup.

The Hilbert space $\mathscr{K}_2$ equals the space of K-invariant vectors in $1_{eK} \cdot (\mathscr{K} \otimes \overline{\mathscr{H}_0}) \cdot 1_{eK}$. In this way, the space of K-invariant vectors in $1_{eK} \cdot (\mathscr{K} \otimes L^2(G/K_0)_h) \cdot 1_{eK}$ naturally is a subspace of $\mathscr{K}_2$. But this last space of K-invariant vectors can be unitarily identified with $\Psi(1_{Kh^{-1}}\, p_{hK_0h^{-1}})\mathscr{K}$ by sending the vector $\xi_0 \in \Psi(1_{Kh^{-1}}\, p_{hK_0h^{-1}})\mathscr{K}$ to the vector

$$\Delta(h)^{-1/2} \sum_{k \in K/hK_0h^{-1}} \Psi(k)\xi_0 \otimes 1_{khK_0} \in \mathscr{K} \otimes L^2(G/K_0)\ .$$

We now use that for every $\mathscr{H} \in \mathscr{C}$, the categorical trace $\mathrm{Tr}_{\mathscr{H}}$ on $(\mathscr{H}, \mathscr{H})$ is given by

$$\begin{aligned}\mathrm{Tr}_{\mathscr{H}}(V) &= \sum_{x \in G/K, \eta \in \mathrm{onb}(1_{xK} \cdot \mathscr{H} \cdot 1_{eK})} \Delta(x)^{-1/2} \langle V\eta, \eta\rangle \\ &= \sum_{y \in G/K, \eta \in \mathrm{onb}(1_{eK} \cdot \mathscr{H} \cdot 1_{yK})} \Delta(y)^{-1/2} \langle V\eta, \eta\rangle\ .\end{aligned}$$

A straightforward computation then gives that for all $\mathscr{H} \in \mathscr{C}$ and all

$$V \in \left(\, \overline{L^2(G/K_0)_g}\ \mathscr{H}\ ,\ \mathscr{H}\ \overline{L^2(G/K_1)_h}\, \right)$$

with $g, h \in G$ and $K_0 < K \cap g^{-1}Kg$, $K_1 < K \cap h^{-1}Kh$ open subgroups, we have

$$\begin{aligned}\Theta(V) = \Delta(g)^{-1/2}\, \Delta(h)^{1/2}\, [K : K_1] & \sum_{\substack{x \in G/gK_0g^{-1} \\ y \in K/K_2 \\ \eta \in \mathrm{onb}(1_{xK} \cdot \mathscr{H} \cdot 1_{eK})}} \Delta(x)^{-1/2} \\ & \Psi(1_{K_2y^{-1}h^{-1}}\, u_x\, p_{gK_0g^{-1}})\ \langle \widetilde{V}(1_{xgK_0} \otimes \eta), \pi_{\mathscr{H}}(hy)\eta \otimes 1_{hK_1}\rangle\ ,\end{aligned} \tag{3.7}$$

whenever $K_2 < K$ is a small enough open subgroup such that $\pi_{\mathscr{H}}(k)$ is the identity on $\mathscr{H} \cdot 1_{eK}$ for all $k \in K_2$. Note that because $\mathscr{H}$ has finite rank, such an open subgroup K_2 exists. Also, there are only finitely many $x \in G/K$ such that $1_{xK} \cdot \mathscr{H} \cdot 1_{eK}$ is nonzero. Therefore, the sum appearing in (3.7) is finite.

Applying this to the regular representation $C_0(G) \rtimes^f_{\mathrm{Ad}} G \to B(L^2(G \times G))$, we see that (3.7) provides a $*$-anti-homomorphisms Θ from $\mathscr{A}$ to $\mathrm{Pol}(L^\infty(G) \rtimes_{\mathrm{Ad}} G)$. Then, a direct computation gives that Θ is trace preserving, using the trace Tr on $L^\infty(G) \rtimes_{\mathrm{Ad}} G$ defined in (3.2). In particular, Θ is injective.

We now prove that Θ is surjective. Fix elements $g, h, \alpha \in G$ satisfying $\alpha g = h\alpha$. Choose any open subgroup $K_0 < K$ such that gK_0g^{-1}, $\alpha K_0 \alpha^{-1}$ and $K_1 := h^{-1}\alpha K_0 \alpha^{-1} h$ are all subgroups of K. Put $\mathscr{H} = L^2(G/K_0)_\alpha$ and note that $\mathscr{H}$, $L^2(G/K_0)_g$ and $L^2(G/K_1)_h$ are well defined objects in $\mathscr{C}$. For every $k \in K$, we

consider the vectors

$$1_{k\alpha g K_0} \otimes 1_{k\alpha K_0} \in 1_{k\alpha g K} \cdot \big(L^2(G/K_0)_g \, \mathscr{H}\big) \cdot 1_{eK} \quad \text{and}$$
$$1_{kh\alpha K_0} \otimes 1_{khK_1} \in 1_{k\alpha g K} \cdot \big(\mathscr{H} \, L^2(G/K_1)_h\big) \cdot 1_{eK} \; .$$

In both cases, we get an orthogonal family of vectors indexed by

$$k \in K/(K \cap \alpha K_0 \alpha^{-1} \cap \alpha g K_0 (\alpha g)^{-1}) \; .$$

So, we can uniquely define $V \in \big(\overline{L^2(G/K_0)_g} \; \mathscr{H} \; , \; \mathscr{H} \; \overline{L^2(G/K_1)_h} \; \big)$ such that the restriction of $\widetilde{V}$ to $\big(L^2(G/K_0)_g \, \mathscr{H}\big) \cdot 1_{eK}$ is the partial isometry given by

$$1_{k\alpha g K_0} \otimes 1_{k\alpha K_0} \mapsto \Delta(\alpha)^{-1/2} \, \Delta(h)^{-1/2} \, 1_{kh\alpha K_0} \otimes 1_{khK_1} \quad \text{for all} \;\; k \in K \; .$$

A direct computation gives that $\Theta(V)$ is equal to a nonzero multiple of

$$1_{\alpha K_0 \alpha^{-1} h^{-1}} \, u_\alpha \, p_{g K_0 g^{-1}} \; . \tag{3.8}$$

From (3.7), we also get that Θ maps $(\mathscr{H}_0, \mathscr{H}_0) \subset \mathscr{A}$ onto $\mathrm{Pol}(L^\infty(K\backslash G) \rtimes K)$, defined as the linear span of all

$$1_{Kx} \, u_k \, p_L$$

with $x \in G$, $k \in K$ and $L < K$ an open subgroup. In combination with (3.8), it follows that Θ is surjective.

Finally, by restricting (3.7) to the cases where $g = h = e$ and $K_0 = K_1 = K$, we find that (3.5) holds. □

We recall from [26] the notion of a *completely positive (cp) multiplier* on a rigid C*-tensor category $\mathscr{C}$. By [26, Proposition 3.6], to every function $\varphi : \mathrm{Irr}(\mathscr{C}) \to \mathbb{C}$ is associated a system of linear maps

$$\Psi^\varphi_{\alpha_1|\beta_1,\alpha_2|\beta_2} : (\alpha_1\beta_1, \alpha_2\beta_2) \to (\alpha_1\beta_1, \alpha_2\beta_2) \quad \text{for all} \;\; \alpha_i, \beta_i \in \mathscr{C} \tag{3.9}$$

satisfying

$$\Psi^\varphi_{\alpha_3|\beta_3,\alpha_4|\beta_4}((X \otimes Y)V(Z \otimes T)) = (X \otimes Y) \; \Psi^\varphi_{\alpha_1|\beta_1,\alpha_2|\beta_2}(V) \; (Z \otimes T)$$

for all $X \in (\alpha_3, \alpha_1)$, $Y \in (\beta_3, \beta_1)$, $Z \in (\alpha_2, \alpha_4)$, $T \in (\beta_2, \beta_4)$, as well as

$$\Psi^\varphi_{\alpha|\overline{\alpha},\varepsilon|\varepsilon}(s_\alpha) = \varphi(\alpha) \, s_\alpha \quad \text{and}$$
$$\Psi^\varphi_{\alpha_1\alpha_2|\beta_2\beta_1,\alpha_3\alpha_4|\beta_4\beta_3}(1 \otimes V \otimes 1) = 1 \otimes \Psi^\varphi_{\alpha_2|\beta_2,\alpha_4|\beta_4}(V) \otimes 1$$

for all $V \in (\alpha_2\beta_2, \alpha_4\beta_4)$.

Definition 3.2 ([26, Definition 3.4]) Let $\mathscr{C}$ be a rigid C*-tensor category.

- A *cp-multiplier* on $\mathscr{C}$ is a function $\varphi : \mathrm{Irr}(\mathscr{C}) \to \mathbb{C}$ such that the maps $\Psi^{\varphi}_{\alpha|\beta,\alpha|\beta}$ on $(\alpha\beta, \alpha\beta)$ are completely positive for all $\alpha, \beta \in \mathscr{C}$.
- A cp-multiplier $\varphi : \mathrm{Irr}(\mathscr{C}) \to \mathbb{C}$ is said to be c_0 if the function $\varphi : \mathrm{Irr}(\mathscr{C}) \to \mathbb{C}$ tends to zero at infinity.
- A *cb-multiplier* on $\mathscr{C}$ is a function $\varphi : \mathrm{Irr}(\mathscr{C}) \to \mathbb{C}$ such that

$$\|\varphi\|_{\mathrm{cb}} := \sup_{\alpha_i,\beta_i \in \mathscr{C}} \|\Psi^{\varphi}_{\alpha_1|\beta_1,\alpha_2|\beta_2}\|_{\mathrm{cb}} < \infty \, .$$

A function $\varphi : \mathrm{Irr}(\mathscr{C}) \to \mathbb{C}$ gives rise to the following linear functional $\omega_\varphi : \mathscr{A} \to \mathbb{C}$ on the tube algebra $\mathscr{A}$ of $\mathscr{C}$ with respect to any full family of objects containing once the trivial object ε:

$$\omega_\varphi : \mathscr{A} \to \mathbb{C} : \omega_\varphi(V) = \begin{cases} d(\alpha)\,\varphi(\alpha) & \text{if } V = 1_\alpha \in (\varepsilon\alpha, \alpha\varepsilon), \\ 0 & \text{if } V \in (i\alpha, \alpha j) \text{ with } i \neq \varepsilon \text{ or } j \neq \varepsilon. \end{cases}$$

By [9, Theorem 6.6], the function $\varphi : \mathrm{Irr}(\mathscr{C}) \to \mathbb{C}$ is a cp-multiplier in the sense of Definition 3.2 if and only if ω_φ is positive on $\mathscr{A}$ in the sense that $\omega_\varphi(V \cdot V^\#) \geq 0$ for all $V \in \mathscr{A}$. In Proposition 5.1, we prove a characterization of cb-multipliers in terms of completely bounded multipliers of the tube $*$-algebra.

From Theorem 3.1, we then get the following result. We again denote by $\mathscr{C}$ be the rigid C*-tensor category of finite rank G-$L^\infty(G/K)$-$L^\infty(G/K)$-modules and we identify $\mathrm{Irr}(\mathscr{C})$ as in (3.4) with the set of pairs (π, x) where $x \in K\backslash G/K$ and π is an irreducible representation of the compact group $K \cap xKx^{-1}$. In order to identify the c_0 cp-multipliers on $\mathscr{C}$, we introduce the following definition.

Definition 3.3 We say that a complex measure μ on G (i.e. an element of $C_0(G)^*$) is c_0 if

$$\lambda(\mu) := \int_G \lambda_g \, d\mu(g) \in L(G)$$

belongs to $C^*_r(G)$.

We say that a positive functional ω on $C_0(G) \rtimes^f_{\mathrm{Ad}} G$ is c_0 if for every $x \in G$, the complex measure μ_x defined by $\mu_x(F) = \omega(F u_x)$ for all $F \in C_0(G)$ is c_0 and if the function $G \to C^*_r(G) : x \mapsto \lambda(\mu_x)$ tends to zero at infinity, i.e. $\lim_{x\to\infty} \|\lambda(\mu_x)\| = 0$.

Proposition 3.4 *The formula*

$$\varphi(\pi, x) = \omega(p_K \dim(\pi)^{-1} \chi_\pi \, u_x \, p_K) \tag{3.10}$$

gives a bijection between the cp-multipliers φ *on* $\mathrm{Irr}(\mathscr{C})$ *and the positive functionals* ω *on the* C^**-algebra* $q(C_0(G) \rtimes^f_{\mathrm{Ad}} G)q$*, where* $q = 1_K p_K$*.*

The cp-multiplier φ is c_0 if and only if the positive functional ω is c_0 in the sense of Definition 3.3.

Using the notations $C_u(\mathscr{C})$ and $C_r(\mathscr{C})$ of [26, Definition 4.1] for the universal and reduced C^*-algebra of $\mathscr{C}$, we have the natural anti-isomorphisms $C_u(\mathscr{C}) \to q(C_0(G) \rtimes^f_{\mathrm{Ad}} G)q$ and $C_r(\mathscr{C}) \to q(C_0(G) \rtimes^r_{\mathrm{Ad}} G)q$.

Proof Note that the G-$L^\infty(G/K)$-$L^\infty(G/K)$-module $\mathscr{H}_0$ in (3.1) contains exactly once the trivial module. The first part of the proposition is then a direct consequence of Theorem 3.1 and the above mentioned characterization [9] of cp-multipliers as positive functionals on the tube $*$-algebra. The isomorphisms for $C_u(\mathscr{C})$ and $C_r(\mathscr{C})$ follow in the same way.

Fix a positive functional ω on $q(C_0(G) \rtimes^f_{\mathrm{Ad}} G)q$ with corresponding cp-multiplier $\varphi : \mathrm{Irr}(\mathscr{C}) \to \mathbb{C}$ given by (3.10). We extend ω to $C_0(G) \rtimes^f_{\mathrm{Ad}} G$ by $\omega(T) = \omega(qTq)$. For every $x \in G$, define $\mu_x \in C_0(G)^*$ given by $\mu_x(F) = \omega(Fu_x)$ for all $F \in C_0(G)$. Note that μ_x is supported on $K \cap xKx^{-1}$ and that μ_x is $\mathrm{Ad}(K \cap xKx^{-1})$-invariant. Therefore, $\lambda(\mu_x) \in \mathscr{Z}(L(K \cap xKx^{-1}))$. For every $\pi \in \mathrm{Irr}(K \cap xKx^{-1})$, denote by $z_\pi \in \mathscr{Z}(L(K \cap xKx^{-1}))$ the corresponding minimal central projection. From (3.10), we get that

$$\lambda(\mu_x)z_\pi = \varphi(\pi, x)z_\pi \quad \text{for all } \; x \in G \; , \; \pi \in \mathrm{Irr}(K \cap xKx^{-1}) \; . \tag{3.11}$$

For a fixed $x \in G$, an element $T \in \mathscr{Z}(L(K \cap xKx^{-1}))$ belongs to $C^*_r(G)$ if and only if $T \in C^*_r(K \cap xKx^{-1})$ if and only if $\lim_{\pi \to \infty} \|Tz_\pi\| = 0$.

Also, $\|T\| = \sup_{\pi \in \mathrm{Irr}(K \cap xKx^{-1})} \|Tz_\pi\|$. So by (3.11), we get that μ_x is c_0 if and only if

$$\lim_{\pi \to \infty} |\varphi(\pi, x)| = 0 \tag{3.12}$$

and that ω is a c_0 functional if and only if (3.12) holds for all $x \in G$ and we moreover have that

$$\lim_{x \to \infty} \Bigl(\sup_{\pi \in \mathrm{Irr}(K \cap xKx^{-1})} |\varphi(\pi, x)| \Bigr) = 0 \; .$$

Altogether, it follows that ω is a c_0 functional in the sense of Definition 3.3 if and only if φ is a c_0-function. □

For later use, we record the following lemma.

Lemma 3.5 *Let μ be a probability measure on G that is c_0 in the sense of Definition 3.3. Then every complex measure $\omega \in C_0(G)^*$ that is absolutely continuous with respect to μ is still c_0.*

Proof Denote by $C_c(G)$ the space of continuous compactly supported functions on G. Since $C_c(G) \subset L^1(G, \mu)$ is dense, it is sufficient to prove that $F \cdot \mu$ is c_0 for every $F \in C_c(G)$. Denote by $\omega_F \in C^*_r(G)^*$ the functional determined by $\omega_F(\lambda_x) = F(x)$

for all $x \in G$. Denote by $\widehat{\Delta} : C_r^*(G) \to M(C_r^*(G) \otimes C_r^*(G))$ the comultiplication determined by $\widehat{\Delta}(\lambda_x) = \lambda_x \otimes \lambda_x$. Recall that for every $\omega \in C_r^*(G)^*$ and every $T \in C_r^*(G)$, we have that $(\omega \otimes \mathrm{id})\widehat{\Delta}(T) \in C_r^*(G)$. Since

$$\lambda(F \cdot \mu) = (\omega_F \otimes \mathrm{id})\widehat{\Delta}(\lambda(\mu)) ,$$

the lemma is proven. □

4 Haagerup Property and Property (T) for $\mathscr{C}(K < G)$

In Definition 3.2, we already recalled the notion of a cp-multiplier $\varphi : \mathrm{Irr}(\mathscr{C}) \to \mathbb{C}$ on a rigid C*-tensor category $\mathscr{C}$. In terms of cp-multipliers, *amenability* of a rigid C*-tensor category, as defined in [13, 21], amounts to the existence of finitely supported cp-multipliers $\varphi_n : \mathrm{Irr}(\mathscr{C}) \to \mathbb{C}$ that converge to 1 pointwise, see [26, Proposition 5.3]. Following [26, Definition 5.1], a rigid C*-tensor category $\mathscr{C}$ has the *Haagerup property* if there exist c_0 cp-multipliers $\varphi_n : \mathrm{Irr}(\mathscr{C}) \to \mathbb{C}$ that converge to 1 pointwise, while $\mathscr{C}$ has *property (T)* if all cp-multipliers converging to 1 pointwise, must converge to 1 uniformly.

Similarly, when $\mathscr{C}_1$ is a full C*-tensor subcategory of $\mathscr{C}$, we say that $\mathscr{C}_1 \subset \mathscr{C}$ has the *relative property (T)* if all cp-multipliers on $\mathscr{C}$ converging to 1 pointwise, must converge to 1 uniformly on $\mathrm{Irr}(\mathscr{C}_1) \subset \mathrm{Irr}(\mathscr{C})$.

We now turn back to the rigid C*-tensor category $\mathscr{C}$ of finite rank G-$L^\infty(G/K)$-$L^\infty(G/K)$-modules, where G is a totally disconnected group G and $K < G$ is a compact open subgroup. Note that $\mathrm{Rep}\, K$ is a full C*-tensor subcategory of $\mathscr{C}$, consisting of the G-$L^\infty(G/K)$-$L^\infty(G/K)$-modules $\mathscr{H}$ with the property that $1_{xK} \cdot \mathscr{H} \cdot 1_{eK}$ is zero for all $x \notin K$.

Recall from Definition 3.3 the notion of a c_0 complex measure on G. We identify the space of complex measures with $C_0(G)^*$ and we denote by $\mathscr{S}(C_0(G)) \subset C_0(G)^*$ the state space of $C_0(G)$, i.e. the set of probability measures on G.

Theorem 4.1 *Let G be a totally disconnected group and $K < G$ a compact open subgroup. Denote by $\mathscr{C}$ the rigid C*-tensor category of finite rank G-$L^\infty(G/K)$-$L^\infty(G/K)$-modules.*

1. *$\mathscr{C}$ is amenable if and only if G is amenable.*
2. *$\mathscr{C}$ has the Haagerup property if and only if G has the Haagerup property and there exists a sequence of c_0 probability measures $\mu_n \in \mathscr{S}(C_0(G))$ such that $\mu_n \to \delta_e$ weakly* and such that $\|\mu_n \circ \mathrm{Ad}\, x - \mu_n\| \to 0$ uniformly on compact sets of $x \in G$.*
3. *$\mathscr{C}$ has property (T) if and only if G has property (T) and every sequence of $\mathrm{Ad}\, G$-invariant probability measures $\mu_n \in \mathscr{S}(C_0(G))$ that converges to δ_e weakly* must converge in norm.*

4. $\operatorname{Rep} K \subset \mathscr{C}$ *has the relative property (T) if and only if every sequence of probability measures* $\mu_n \in \mathscr{S}(C_0(G))$ *such that* $\mu_n \to \delta_e$ *weakly** *and* $\|\mu_n \circ \operatorname{Ad} x - \mu_n\| \to 0$ *uniformly on compact sets of* $x \in G$ *satisfies* $\|\mu_n - \delta_e\| \to 0$.

Proof Denote by $\epsilon : C_0(G) \rtimes^f_{\mathrm{Ad}} G \to \mathbb{C}$ the character given by

$$\epsilon(F\lambda_f) = F(e) \int_G f(x)\, dx\,.$$

Write $q = 1_K p_K$.

1. Combining Proposition 3.4 and [26, Proposition 5.3], we get that $\mathscr{C}$ is amenable if and only if the canonical $*$-homomorphism from $q(C_0(G) \rtimes^f_{\mathrm{Ad}} G)q$ onto $q(C_0(G) \rtimes^r_{\mathrm{Ad}} G)q$ is an isomorphism. This holds if and only if G is amenable.
2. First assume that $\mathscr{C}$ has the Haagerup property. By Proposition 3.4, we find a sequence of states ω_n on $q(C_0(G) \rtimes^f_{\mathrm{Ad}} G)q$ such that $\omega_n \to \epsilon$ weakly* and such that every ω_n is a c_0 state in the sense of Definition 3.3. For every $x \in G$, define $\mu_n(x) \in C_0(G)^*$ given by $\mu_n(x)(F) = \omega_n(Fu_x)$.

Using the strictly continuous extension of ω_n to $M(C_0(G) \rtimes^f_{\mathrm{Ad}} G)$, we get that $x \mapsto \omega_n(u_x)$ is a sequence of continuous positive definite functions converging to 1 uniformly on compact subsets of G. We claim that for every fixed n, the function $x \mapsto \omega_n(x)$ tends to 0 at infinity. Denote by $\epsilon_K : C^*_r(G) \to \mathbb{C}$ the state given by composing the conditional expectation $C^*_r(G) \to C^*_r(K)$ with the trivial representation $\epsilon : C^*_r(K) \to \mathbb{C}$. Then,

$$\omega_n(x) = \epsilon_K(\lambda(\mu_n(x)))$$

and the claim is proven. So, G has the Haagerup property.

The restriction of ω_n to $C_0(G)$ provides a sequence of c_0 probability measures $\mu_n \in \mathscr{S}(C_0(G))$ such that $\mu_n \to \delta_e$ weakly* and $\|\mu_n \circ \operatorname{Ad} x - \mu_n\| \to 0$ uniformly on compact sets of $x \in G$.

Conversely assume that G has the Haagerup property and that μ_n is such a sequence of probability measures. By restricting μ_n to K, normalizing and integrating $\int_K (\mu_n \circ \operatorname{Ad} k)\, dk$, we may assume that the probability measures μ_n are supported on K and are $\operatorname{Ad} K$-invariant. Fix a strictly positive right K-invariant function $w : G \to \mathbb{R}^+_0$ with $\int_G w(g)\, dg = 1$. Define the probability measures $\tilde{\mu}_n$ on G given by

$$\tilde{\mu}_n = \int_G w(g)\, \mu_n \circ \operatorname{Ad} g\, dg\,.$$

Note that $\tilde{\mu}_n$ is still $\operatorname{Ad} K$-invariant. Also,

$$\lambda(\tilde{\mu}_n) = \int_G w(g)\, \lambda^*_g\, \lambda(\mu_n)\, \lambda_g\, dg$$

so that each $\tilde{\mu}_n$ is a c_0 probability measure.

By construction, for every $x \in G$, the measure $\tilde{\mu}_n \circ \mathrm{Ad}\, x$ is absolutely continuous with respect to $\tilde{\mu}_n$. We denote by $\Delta_n(x)$ the Radon-Nikodym derivative and define the unitary representations

$$\theta_n : G \to \mathscr{U}(L^2(G, \tilde{\mu}_n)) : \theta_n(x)\xi = \Delta_n(x)^{1/2}\, \xi \circ \mathrm{Ad}\, x^{-1} .$$

We also define $\theta_n : C_0(G) \to B(L^2(G, \tilde{\mu}_n))$ given by multiplication operators and we have thus defined a nondegenerate $*$-representation of $C_0(G) \rtimes^f_{\mathrm{Ad}} G$ on $L^2(G, \tilde{\mu}_n)$.

Note that μ_n is absolutely continuous with respect to $\tilde{\mu}_n$. We denote by $\zeta_n \in L^2(G, \tilde{\mu}_n)$ the square root of the Radon-Nikodym derivative of μ_n with respect to $\tilde{\mu}_n$. Since both μ_n and $\tilde{\mu}_n$ are $\mathrm{Ad}\, K$-invariant, we get that $\theta_n(p_K)\zeta_n = \zeta_n$. Since μ_n is supported on K, also ζ_n is supported on K meaning that $\theta(1_K)\zeta_n = \zeta_n$.

Since G has the Haagerup property, we can also fix a unitary representation $\pi : G \to \mathscr{U}(\mathscr{H})$ and a sequence of $\pi(K)$-invariant unit vectors $\xi_n \in \mathscr{H}$ such that $\|\pi(x)\xi_n - \xi_n\| \to 0$ uniformly on compact sets of $x \in G$ and, for every fixed n, the function $x \mapsto \langle \pi(x)\xi_n, \xi_n \rangle$ tends to zero at infinity.

The formulas $\psi(x) = \theta_n(x) \otimes \pi(x)$ and $\psi(F) = \theta(F) \otimes 1$ define a nondegenerate $*$-representation of $C_0(G) \rtimes^f_{\mathrm{Ad}} G$ on $L^2(G, \tilde{\mu}_n) \otimes \mathscr{H}$. We define the states ω_n on $C_0(G) \rtimes^f_{\mathrm{Ad}} G$ given by $\omega_n(T) = \langle \psi(T)(\zeta_n \otimes \xi_n), \zeta_n \otimes \xi_n \rangle$. By construction, $\omega_n(q) = 1$ for all n and $\omega_n \to \epsilon$ weakly*. It remains to prove that each ω_n is a c_0 state. Proposition 3.4 then gives that $\mathscr{C}$ has the Haagerup property.

Fix n. Defining $\mu_n(x) \in C_0(G)^*$ given by $\mu_n(x)(F) = \omega_n(Fu_x)$, we get that

$$\mu_n(x)(F) = \langle \theta_n(F)\, \theta(x)\, \zeta_n, \zeta_n \rangle \, \langle \pi(x)\xi_n, \xi_n \rangle \, .$$

Since the function $x \mapsto \langle \pi(x)\xi_n, \xi_n \rangle$ tends to zero at infinity, we get that even the function $x \mapsto \|\mu_n(x)\|$ tends to zero at infinity. So, we only have to show that for every fixed x, the complex measure given by $F \mapsto \langle \theta_n(F)\, \theta(x)\, \zeta_n, \zeta_n \rangle$ is c_0. By construction, this complex measure is absolutely continuous with respect to $\tilde{\mu}_n$. The conclusion then follows from Lemma 3.5.

3. Note that it follows from [26, Proposition 5.5] that $\mathscr{C}$ has property (T) if and only if every sequence of states on $q(C_0(G) \rtimes^f_{\mathrm{Ad}} G)q$ converging weakly* to ϵ must converge to ϵ in norm.

First assume that $\mathscr{C}$ has property (T). Both states on $C^*(G)$ and $\mathrm{Ad}\, G$-invariant states on $C_0(G)$ give rise to states on $C_0(G) \rtimes^f_{\mathrm{Ad}} G$. One implication of 3 thus follows immediately. Conversely assume that G has property (T) and that every sequence of $\mathrm{Ad}\, G$-invariant probability measures $\mu_n \in \mathscr{S}(C_0(G))$ converging weakly* to δ_e must converge in norm to δ_e. Let ω_n be a sequence of states on $q(C_0(G) \rtimes^f_{\mathrm{Ad}} G)q$ converging to ϵ weakly*. Let $p \in C^*(G)$ be the Kazhdan projection. Replacing ω_n by $\omega_n(p)^{-1}\, p \cdot \omega_n \cdot p$, we may assume that ω_n is left and right G-invariant. This means that $\omega_n(Fu_x) = \mu_n(F)$ for all $F \in C_0(G)$, $x \in G$, where μ_n is a sequence of $\mathrm{Ad}\, G$-invariant probability measures on G converging weakly* to δ_e. Thus $\|\mu_n - \delta_e\| \to 0$ so that $\|\omega_n - \epsilon\| \to 0$.

4. First assume that $\operatorname{Rep} K \subset \mathscr{C}$ has the relative property (T) and take a sequence of probability measures $\mu_n \in \mathscr{S}(C_0(G))$ such that $\mu_n \to \delta_e$ weakly* and such that $\|\mu_n \circ \operatorname{Ad} x - \mu_n\| \to 0$ uniformly on compact sets of $x \in G$. We must prove that $\|\mu_n - \delta_e\| \to 0$. As in the proof of 2, we may assume that μ_n is supported on K and that μ_n is $\operatorname{Ad} K$-invariant, so that we can construct a sequence of states ω_n on $C_0(G) \rtimes^f_{\mathrm{Ad}} G$ such that $\omega_n \to \delta_e$ weakly*, $\omega_n = q \cdot \omega_n \cdot q$ and $\omega_n|_{C_0(G)} = \mu_n$ for all n.

The formula (3.10) associates to ω_n a sequence of cp-multipliers φ_n on $\mathscr{C}$ converging to 1 pointwise. Since $\operatorname{Rep} K \subset \mathscr{C}$ has the relative property (T), we conclude that $\varphi_n(\pi, e) \to 1$ uniformly on $\pi \in \operatorname{Irr}(K)$. Using [26, Lemma 5.6], it follows that $\|\omega_n|_{C_0(G)} - \delta_e\| \to 0$. So, $\|\mu_n - \delta_e\| \to 0$.

To prove the converse, let $\varphi_n : \operatorname{Irr}(\mathscr{C}) \to \mathbb{C}$ be a sequence of cp-multipliers on $\mathscr{C}$ converging to 1 pointwise. Denote by ω_n the states on $q(C_0(G) \rtimes^f_{\mathrm{Ad}} G)q$ associated with φ_n in Proposition 3.4. Since $\omega_n \to \epsilon$ weakly*, the restriction $\mu_n := \omega_n|_{C_0(G)}$ is a sequence of probability measures on G such that $\mu_n \to \delta_e$ weakly* and $\|\mu_n \circ \operatorname{Ad} x - \mu_n\| \to 0$ uniformly on compact sets of $x \in G$. By our assumption, $\|\mu_n - \delta_e\| \to 0$. For every $\pi \in \operatorname{Irr}(K)$, the function $\dim(\pi)^{-1}\chi_\pi$ has norm 1. Therefore, $\omega_n(\dim(\pi)^{-1}\chi_\pi) \to 1$ uniformly on $\operatorname{Irr}(K)$. By (3.10), this means that $\varphi_n \to 1$ uniformly on $\operatorname{Irr}(K)$. ◻

The following proposition gives a concrete example where G has the Haagerup property, while $\mathscr{C}(K < G)$ does not and even has $\operatorname{Rep} K$ as a full C*-tensor subcategory with the relative property (T).

Proposition 4.2 *Let F be a non-Archimedean local field with characteristic $\neq 2$. Let $k \geq 2$ and define $G = \mathrm{SL}(k, F)$. Let $K < G$ be a compact open subgroup, e.g. $K = \mathrm{SL}(k, \mathscr{O})$, where $\mathscr{O}$ is the ring of integers of F. Denote by $\mathscr{C}$ the rigid C*-tensor category of finite rank G-$L^\infty(G/K)$-$L^\infty(G/K)$-modules.*

1. $\operatorname{Rep} K \subset \mathscr{C}$ has the relative property (T). In particular, $\mathscr{C}$ does not have the Haagerup property, although for $k = 2$, the group G has the Haagerup property.
2. $\mathscr{C}$ has property (T) for all $k \geq 3$.

Proof We denote by $\mathbb{I}$ the identity element of $G = \mathrm{SL}(k, F)$. Let $\mu_n \in \mathscr{S}(C_0(G))$ be a sequence of probability measures on G such that $\mu_n \to \delta_{\mathbb{I}}$ weakly* and such that $\|\mu_n \circ \operatorname{Ad} x - \mu_n\| \to 0$ uniformly on compact sets of $x \in G$. For contradiction, assume that $\|\mu_n - \delta_{\mathbb{I}}\| \not\to 0$. Passing to a subsequence and replacing μ_n by the normalization of $\mu_n - \mu_n(\{\mathbb{I}\})\delta_{\mathbb{I}}$, we may assume that $\mu_n(\{\mathbb{I}\}) = 0$ for all n. Since $\mu_n \to \delta_{\mathbb{I}}$ weakly* and since there are at most k of k'th roots of unity in F, we may also assume that $\mu_n(\{\lambda\mathbb{I}\}) = 0$ for all n and all k'th roots of unity $\lambda \in F$.

Every μ_n defines a state Ω_n on the C*-algebra $\mathscr{L}(G)$ of all bounded Borel functions on G. Choose a weak*-limit point $\Omega \in \mathscr{L}(G)^*$ of the sequence (Ω_n). Then, Ω induces an $\operatorname{Ad} G$-invariant mean on the Borel sets of G. In particular, Ω defines an $\operatorname{Ad} G$-invariant mean Ω on the Borel sets of the space $M_n(F)$ of $n \times n$ matrices over F. By Lemma 4.5 below, Ω is supported on the diagonal $F\mathbb{I} \subset M_n(F)$. Since Ω is also supported on G, it follows that Ω is supported on the finite set of $\lambda\mathbb{I}$

where λ is a k'th root of unity in F. But by construction, $\Omega(\{\lambda\mathbb{I}\}) = 0$ for all k'th roots of unity $\lambda \in F$. We have reached a contradiction. So, $\|\mu_n - \delta_{\mathbb{I}}\| \to 0$.

By Theorem 4.1, $\operatorname{Rep} K \subset \mathscr{C}$ has the relative property (T). For $k \geq 3$, the group $\mathrm{SL}(k, F)$ has property (T) and it follows from Theorem 4.1 that $\mathscr{C}$ has property (T). $\square$

The following example of [8] illustrates that G may have property (T), while the category $\mathscr{C}$ of finite rank G-$L^\infty(G/K)$-$L^\infty(G/K)$-modules does not.

Example 4.3 Let F be a non-Archimedean local field and $k \geq 3$. Define the closed subgroup $G < \mathrm{SL}(k+2, F)$ given by

$$G = \left\{ \begin{pmatrix} 1 & b_1 & \cdots & b_k & c \\ 0 & a_{11} & \cdots & a_{1k} & d_1 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & a_{k1} & \cdots & a_{kk} & d_k \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \middle| A = (a_{ij}) \in \mathrm{SL}(k, F)\ ,\ b_i, c, d_j \in F \right\} .$$

As in [8], we get that G has property (T). Also, the center of G is isomorphic with F (sitting in the upper right corner) and since F is non discrete, we can take a sequence $g_n \in \mathscr{Z}(G)$ with $g_n \neq e$ for all n and $g_n \to e$. Using the $\operatorname{Ad} G$-invariant probability measures δ_{g_n}, it follows from Theorem 4.1 that $\mathscr{C}$ does not have property (T).

Finally, we also include a nonamenable example having the Haagerup property.

Example 4.4 Let $2 \leq |m| < n$ be integers. Define the totally disconnected compact abelian group $K = \mathbb{Z}_{nm}$ as the profinite completion of $\mathbb{Z}$ with respect to the decreasing sequence of finite index subgroups $(n^k m^k \mathbb{Z})_{k \geq 0}$. We have open subgroups $mK < K$ and $nK < K$, as well as the isomorphism $\varphi : mK \to nK : \varphi(mk) = nk$ for all $k \in K$. We define G as the HNN extension of K and φ. Alternatively, we may view $K < G$ as the Schlichting completion of the Baumslag-Solitar group

$$B(m, n) = \langle a, t \mid t a^m t^{-1} = a^n \rangle$$

and the almost normal subgroup $\langle a \rangle$.

Since G is acting properly on a tree, G has the Haagerup property. Also, G is nonamenable. For all positive integers $k, l \geq 0$, we denote by $\mu_{k,l}$ the normalized Haar measure on the open subgroup $n^k m^l K$. Note that $\varphi_*(\mu_{k,l}) = \mu_{k+1,l-1}$ whenever $k, l \geq 1$. Then the probability measures

$$\mu_n := \frac{1}{n+1} \sum_{k=0}^{n} \mu_{n+k, 2n-k}$$

are absolutely continuous with respect to the Haar measure of G, and thus c_0 in the sense of Definition 3.3, and they satisfy $\mu_n \to \delta_e$ weakly* and $\|\mu_n \circ \operatorname{Ad} x - \mu_n\| \to 0$ uniformly on compact sets of $x \in G$. By Theorem 4.1, $\mathscr{C}$ has the Haagerup property.

Lemma 4.5 *Let F be a local field with characteristic $\neq 2$. Let $k \geq 2$ and define $G = \mathrm{SL}(k, F)$. Every $\operatorname{Ad} G$-invariant mean on the Borel sets of the space $M_k(F)$ of $k \times k$ matrices over F is supported on the diagonal $F\mathbb{I} \subset M_k(F)$.*

Proof We start by proving the lemma for $k = 2$. So assume that m is an $\operatorname{Ad}\mathrm{SL}(2, F)$-invariant mean on the Borel sets of $M_2(F)$.

In the proof of [2, Proposition 1.4.12], it is shown that if m is a mean on the Borel sets of F^2 that is invariant under the transformations $\lambda \cdot (x, y) := (x + \lambda y, y)$ for all $\lambda \in F$, then

$$m(\{(x, y) \mid (x, y) \neq (0, 0), |x| \leq |y|\}) = 0 .$$

Define $g_\lambda := \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}$ and notice that

$$g_\lambda \begin{pmatrix} a & b \\ c & d \end{pmatrix} g_\lambda^{-1} = \begin{pmatrix} a + \lambda c & -\lambda a + b - \lambda^2 c + \lambda d \\ c & -\lambda c + d \end{pmatrix} .$$

Hence, the map $\theta : M_2(F) \to F^2 : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (a - d, c)$ satisfies $\theta(g_\lambda A g_\lambda^{-1}) = (2\lambda) \cdot \theta(A)$. Therefore, $m(\Omega_0) = 0$ for

$$\Omega_0 := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(F) \;\middle|\; |a - d| \leq |c| \text{ and } (a - d, c) \neq (0, 0) \right\} .$$

Taking the adjoint by $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}$ for $|\lambda| \geq 2$, we get that $m(\Omega_1) = 0$ for

$$\Omega_1 := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(F) \;\middle|\; |a - d| \leq 4|c| \text{ and } (a - d, c) \neq (0, 0) \right\} .$$

For the same reason, we get that $m(\Omega_1') = 0$ for

$$\Omega_1' := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(F) \;\middle|\; |a - d| \leq 4|b| \text{ and } (a - d, b) \neq (0, 0) \right\} .$$

Write $X = M_2(F) \setminus F\mathbb{I}$. The matrices with $(a - d, c) = (0, 0)$ belong to Ω_1' unless they are diagonal. Similarly, the matrices with $(a - d, b) = (0, 0)$ belong to Ω_1

unless they are diagonal. So, we find that $m(\Omega) = 0 = m(\Omega')$ for

$$\Omega = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in X \;\middle|\; |a-d| \leq 4|c| \right\} \quad \text{and} \quad \Omega' = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in X \;\middle|\; |a-d| \leq 4|b| \right\} .$$

Put $\Omega'' := g_1 \Omega g_1^{-1}$, so that $m(\Omega'') = 0$. To conclude the proof in the case $k = 2$, it suffices to show that $\Omega \cup \Omega' \cup \Omega'' = X$.

Take $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in X \setminus (\Omega \cup \Omega')$. So, $\frac{1}{4}|a-d| > |b|, |c|$. We claim that

$$\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} := g_1^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} g_1 = \begin{pmatrix} a-c & a+b-c-d \\ c & c+d \end{pmatrix}$$

belongs to Ω. Since

$$|a'-d'| = |a-d-2c| \leq |a-d| + 2|c| < \frac{3}{2}|a-d| \quad \text{and}$$

$$|b'| \geq |a-d| - |c| - |b| > \frac{1}{2}|a-d| ,$$

we indeed get that $|a'-d'| < 3|b'|$. The claim follows and the lemma is proved in the case $k = 2$.

For an arbitrary $k \geq 2$ and fixed $1 \leq p < q \leq k$, the map

$$M_k(F) \to M_2(F) : (x_{ij}) \mapsto \begin{pmatrix} x_{pp} & x_{pq} \\ x_{qp} & x_{qq} \end{pmatrix}$$

is $\operatorname{Ad} \mathrm{SL}(2,F)$-equivariant. So, an $\operatorname{Ad} \mathrm{SL}(k,F)$-invariant mean m on $M_k(F)$ is supported on $\{(x_{ij}) \in M_k(F) \mid x_{pp} = x_{qq}, x_{pq} = x_{qp} = 0\}$. Since $F\mathbb{I}$ is the intersection of these sets, m is supported on $F\mathbb{I}$. ☐

5 Weak Amenability of Rigid C*-Tensor Categories

Following [26, Definition 5.1], a rigid C*-tensor category is called *weakly amenable* if there exists a sequence of completely bounded (cb) multipliers $\varphi_n : \mathrm{Irr}(\mathscr{C}) \to \mathbb{C}$ (see Definition 3.2) converging to 1 pointwise, with $\limsup_n \|\varphi_n\|_{\mathrm{cb}} < \infty$ and with φ_n finitely supported for every n.

Recall from the first paragraphs of Sect. 3 the definition of the tube $*$-algebra $\mathscr{A}$ of $\mathscr{C}$ with respect to a full family of objects in $\mathscr{C}$. To every function $\varphi : \mathrm{Irr}(\mathscr{C}) \to \mathbb{C}$, we associate the linear map

$$\theta_\varphi : \mathscr{A} \to \mathscr{A} : \theta_\varphi(V) = \varphi(\alpha)\, V \quad \text{for all} \;\; V \in (i\alpha, \alpha j) .$$

We define $\|\theta_\varphi\|_{\mathrm{cb}}$ by viewing $\mathscr{A}$ inside its reduced C*-algebra, i.e. by viewing $\mathscr{A} \subset B(L^2(\mathscr{A}, \mathrm{Tr}))$, where Tr is the canonical trace on $\mathscr{A}$. We also consider the von Neumann algebra $\mathscr{A}''$ generated by $\mathscr{A}$ acting on $L^2(\mathscr{A}, \mathrm{Tr})$.

In the following result, we clarify the link between the complete boundedness of φ in the sense of Definition 3.2 and the complete boundedness of the map θ_φ.

Proposition 5.1 *Let $\mathscr{C}$ be a rigid C*-tensor category. Denote by $\mathscr{A}$ the tube $*$-algebra of $\mathscr{C}$ with respect to a full family of objects in $\mathscr{C}$. Let $\varphi : \mathrm{Irr}(\mathscr{C}) \to \mathbb{C}$ be any function.*

Then, $\|\varphi\|_{\mathrm{cb}} = \|\theta_\varphi\|_{\mathrm{cb}}$. If this cb-norm is finite, we can uniquely extend θ_φ to a normal completely bounded map on $\mathscr{A}''$ having the same cb-norm.

Proof For any family J of objects, we can define the tube $*$-algebra $\mathscr{A}_J$ and the linear map $\theta_\varphi^J : \mathscr{A}_J \to \mathscr{A}_J$. By strong Morita equivalence, we have $\|\theta_\varphi^J\|_{\mathrm{cb}} = \|\theta_\varphi\|_{\mathrm{cb}}$ whenever J is full and we have $\|\theta_\varphi^J\|_{\mathrm{cb}} \leq \|\theta_\varphi\|_{\mathrm{cb}}$ for arbitrary J. Also, using standard solutions for the conjugate equations, we get natural linear maps $(i\alpha, \alpha j) \to (\overline{j}\overline{\alpha}, \overline{\alpha}\overline{i})$ and they define a trace preserving $*$-anti-isomorphism of $\mathscr{A}_J$ onto $\mathscr{A}_{\overline{J}}$. Defining $\tilde{\varphi} : \mathrm{Irr}(\mathscr{C}) \to \mathbb{C}$ by $\tilde{\varphi}(\alpha) = \varphi(\overline{\alpha})$ for all $\alpha \in \mathrm{Irr}(\mathscr{C})$, it follows that $\|\theta_\varphi\|_{\mathrm{cb}} = \|\theta_{\tilde{\varphi}}\|_{\mathrm{cb}}$ and it follows that θ_φ extends to a normal completely bounded map on $\mathscr{A}''$ if and only if $\theta_{\tilde{\varphi}}$ extends to $\mathscr{A}''$.

So, it suffices to prove that $\|\varphi\|_{\mathrm{cb}} = \|\theta_{\tilde{\varphi}}\|_{\mathrm{cb}}$ and that in the case where $\|\varphi\|_{\mathrm{cb}} < \infty$, we can extend $\theta_{\tilde{\varphi}}$ to a normal completely bounded map on $\mathscr{A}''$. First assume that $\|\theta_{\tilde{\varphi}}\|_{\mathrm{cb}} \leq \kappa$. Fix arbitrary objects $\alpha, \beta \in \mathscr{C}$ and write $\Psi^\varphi_{\alpha|\beta} := \Psi^\varphi_{\alpha|\beta,\alpha|\beta}$. We prove that $\|\Psi^\varphi_{\alpha|\beta}\|_{\mathrm{cb}} \leq \kappa$. Since α, β were arbitrary, it then follows that $\|\varphi\|_{\mathrm{cb}} \leq \kappa$.

Note that $(\alpha\beta, \alpha\beta)$ is a finite dimensional C*-algebra. Consider the following three bijective linear maps, making use of standard solutions of the conjugate equations.

$$\eta_1 : \bigoplus_{\pi \in \mathrm{Irr}(\mathscr{C})} \big((\alpha, \alpha\pi) \otimes (\pi\beta, \beta)\big) \to (\alpha\beta, \alpha\beta) : \eta_1(V \otimes W) = (V \otimes 1)(1 \otimes W) \;\; ,$$

$$\eta_2 : \bigoplus_{\pi \in \mathrm{Irr}(\mathscr{C})} \big((\alpha, \alpha\pi) \otimes (\pi\beta, \beta)\big) \to \bigoplus_{\pi \in \mathrm{Irr}(\mathscr{C})} \big((\alpha\overline{\pi}, \alpha) \otimes (\beta, \overline{\pi}\beta)\big) :$$

$$\eta_2(V \otimes W) = (V \otimes 1)(1 \otimes s_\pi) \otimes (t_\pi^* \otimes 1)(1 \otimes W) \;\; ,$$

$$\eta_3 : \bigoplus_{\pi \in \mathrm{Irr}(\mathscr{C})} \big((\alpha\overline{\pi}, \alpha) \otimes (\beta, \overline{\pi}\beta)\big) \to \mathscr{A}_{\beta\alpha} : \eta_3(V \otimes W) = (1 \otimes V)(W \otimes 1) \;\; .$$

A direct computation shows that $\eta := \eta_3 \circ \eta_2 \circ \eta_1^{-1}$ is a unital faithful $*$-homomorphism of $(\alpha\beta, \alpha\beta)$ to the tube $*$-algebra $\mathscr{A}_{\beta\alpha}$. One also checks that $\theta_{\tilde{\varphi}}^{\beta\alpha} \circ \eta = \eta \circ \Psi^\varphi_{\alpha|\beta}$. So, we get that

$$\|\Psi^\varphi_{\alpha|\beta}\|_{\mathrm{cb}} \leq \|\theta_{\tilde{\varphi}}^{\beta\alpha}\|_{\mathrm{cb}} \leq \|\theta_{\tilde{\varphi}}\|_{\mathrm{cb}} \leq \kappa \;\; .$$

Conversely, assume that $\|\varphi\|_{\mathrm{cb}} \leq \kappa$. Define the ind-objects ρ_1 and ρ_2 for $\mathscr{C}$ given by

$$\rho_1 = \bigoplus_{\alpha,i\in\mathrm{Irr}(\mathscr{C})} \alpha i \quad \text{and} \quad \rho_2 = \bigoplus_{\alpha\in\mathrm{Irr}(\mathscr{C})} \alpha \; .$$

Define the type I von Neumann algebra $\mathscr{M}$ of all bounded endomorphisms of $\rho_1\rho_2$. Note that for all $\alpha, i, \beta \in \mathrm{Irr}(\mathscr{C})$, we have the natural projection $p_\alpha \otimes p_i \otimes p_\beta \in \mathscr{M}$ and we have the identification

$$(p_\alpha \otimes p_i \otimes p_\beta)\mathscr{M}(p_\gamma \otimes p_j \otimes p_\delta) = (\alpha i\beta, \gamma j\delta)$$

for all $\alpha, i, \beta, \gamma, j, \delta \in \mathrm{Irr}(\mathscr{C})$. By our assumption, there is a normal completely bounded map $\Psi : \mathscr{M} \to \mathscr{M}$ satisfying

$$\Psi(V) = \Psi^{\varphi}_{\alpha i|\beta,\gamma j|\delta}(V) \quad \text{for all} \;\; V \in (\alpha i\beta, \gamma j\delta) \; .$$

We have $\|\Psi\|_{\mathrm{cb}} \leq \kappa$.

Consider the projection $q \in \mathscr{M}$ given by

$$q = \sum_{\alpha,i\in\mathrm{Irr}(\mathscr{C})} p_{\overline{\alpha}} \otimes p_i \otimes p_\alpha \; .$$

Since $\Psi(qTq) = q\Psi(T)q$ for all $T \in \mathscr{M}$, the map Ψ restricts to a normal completely bounded map on $q\mathscr{M}q$ with $\|\Psi|_{q\mathscr{M}q}\|_{\mathrm{cb}} \leq \kappa$.

Denote by $\mathscr{A}$ the tube $*$-algebra associated with $\mathrm{Irr}(\mathscr{C})$ itself as a full family of objects. We construct a faithful normal $*$-homomorphism $\Theta : \mathscr{A}'' \to q\mathscr{M}q$ satisfying $\Psi \circ \Theta = \Theta \circ \theta_{\tilde{\varphi}}$. Once we have obtained Θ, it follows that $\|\theta_{\tilde{\varphi}}\|_{\mathrm{cb}} \leq \kappa$ and that $\theta_{\tilde{\varphi}}$ extends to a normal completely bounded map on $\mathscr{A}''$.

To construct Θ, define the Hilbert space

$$\mathscr{H} = \bigoplus_{\alpha,i,j\in\mathrm{Irr}(\mathscr{C})} (\overline{\alpha} i\alpha, j)$$

and observe that we have the natural faithful normal $*$-homomorphism $\pi : q\mathscr{M}q \to B(\mathscr{H})$ given by left multiplication. Also consider the unitary operator

$$U : L^2(\mathscr{A}, \mathrm{Tr}) \to \mathscr{H} : U(V) = d(\alpha)^{-1/2}\,(1 \otimes V)(t_\alpha \otimes 1) \quad \text{for all} \;\; V \in (i\alpha, \alpha j) \; .$$

We claim that Θ can be constructed such that $\pi(\Theta(V)) = UVU^*$ for all $V \in \mathscr{A}$. To prove this claim, fix $i, \alpha, j \in \mathrm{Irr}(\mathscr{C})$ and $V \in (i\alpha, \alpha j)$. For all $\gamma, \beta \in \mathrm{Irr}(\mathscr{C})$, define

the element $W_{\gamma,\beta} \in (\overline{\gamma} i \gamma, \overline{\beta} j \beta)$ given by the finite sum

$$W_{\gamma,\beta} = \sum_{Z \in \mathrm{onb}(\overline{\gamma}\alpha,\overline{\beta})} d(\beta)^{1/2}\, d(\gamma)^{1/2}\, (1 \otimes 1 \otimes \widetilde{Z})\, (1 \otimes V \otimes 1)\, (Z \otimes 1 \otimes 1)\ , \tag{5.1}$$

where $\widetilde{Z} = (1 \otimes t_\beta^*)(1 \otimes Z^* \otimes 1)(s_\gamma \otimes 1)$ belongs to $(\gamma, \alpha\beta)$. A direct computation shows that

$$\langle \pi(W_{\gamma,\beta})\, U(X), U(Y) \rangle = \langle V \cdot X, Y \rangle$$

for all $X \in (j\beta, \beta k)$ and $Y \in (i\gamma, \gamma l)$. So, there is a unique element $\Theta(V)$ that belongs to $(1 \otimes p_i \otimes 1) q \mathscr{M} q (1 \otimes p_j \otimes 1)$ and satisfies

$$(p_{\overline{\gamma}} \otimes p_i \otimes p_\gamma)\, \Theta(V)\, (p_{\overline{\beta}} \otimes p_j \otimes p_\beta) = W_{\gamma,\beta}$$

for all $\gamma, \beta \in \mathrm{Irr}(\mathscr{C})$ and $\pi(\Theta(V)) = UVU^*$.

We have defined a faithful normal $*$-homomorphism $\Theta : \mathscr{A}'' \to q\mathscr{M}q$. It remains to prove that $\Psi \circ \Theta = \Theta \circ \theta_{\tilde{\varphi}}$. Using (5.1), it suffices to prove that

$$\Psi^{\varphi}_{\overline{\gamma} i|\alpha\beta,\overline{\gamma}\alpha j|\beta}(1 \otimes V \otimes 1) = \varphi(\overline{\alpha})\, 1 \otimes V \otimes 1\ . \tag{5.2}$$

The left hand side of (5.2) equals $1 \otimes \Psi^{\varphi}_{i|\alpha,\alpha j|\varepsilon}(V) \otimes 1$. Since $V \in (i\alpha, \alpha j)$, we can uniquely write $V = (T \otimes 1)(1 \otimes 1 \otimes s_{\overline{\alpha}})$ with $T \in (i, \alpha j \overline{\alpha})$. We then have

$$\begin{aligned}\Psi^{\varphi}_{i|\alpha,\alpha j|\varepsilon}(V) &= (T \otimes 1)\, \Psi^{\varphi}_{\alpha j\overline{\alpha}|\alpha,\alpha j|\varepsilon}(1 \otimes 1 \otimes s_{\overline{\alpha}}) = (T \otimes 1)(1 \otimes 1 \otimes \Psi^{\varphi}_{\overline{\alpha}|\alpha,\varepsilon|\varepsilon}(s_{\overline{\alpha}}) \\ &= \varphi(\overline{\alpha})\, (T \otimes 1)(1 \otimes 1 \otimes s_{\overline{\alpha}}) = \varphi(\overline{\alpha})\, V\ .\end{aligned}$$

So (5.2) holds and the proposition is proven. ◻

6 Weak Amenability of $\mathscr{C}(K < G)$

Theorem 6.1 *Let G be a totally disconnected group and $K < G$ a compact open subgroup. Denote by $\mathscr{C}$ the rigid C*-tensor category of finite rank G-$L^\infty(G/K)$-$L^\infty(G/K)$-modules.*

Then $\mathscr{C}$ is weakly amenable if and only if G is weakly amenable and there exists a sequence of probability measures $\omega_n \in C_0(G)^*$ that are absolutely continuous with respect to the Haar measure and such that $\omega_n \to \delta_e$ weakly and $\|\omega_n \circ \mathrm{Ad}\, x - \omega_n\| \to 0$ uniformly on compact sets of $x \in G$.*

In that case, the Cowling-Haagerup constant $\Lambda(\mathscr{C})$ of $\mathscr{C}$ equals $\Lambda(G)$.

In order to prove Theorem 6.1, we must describe the cb-multipliers on $\mathscr{C}$ in terms of completely bounded multipliers on the C*-algebra $C_0(G) \rtimes^r_{\mathrm{Ad}} G$.

We denote by $\operatorname{Pol}(G)$ the $*$-algebra of locally constant, compactly supported functions on G. Note that $\operatorname{Pol}(G)$ is the linear span of the functions of the form 1_{Ly} where $y \in G$ and $L < G$ is a compact open subgroup. Also note that for any compact open subgroup $K_0 < G$, $\operatorname{Pol}(K_0)$ coincides with the $*$-algebra of coefficients of finite dimensional unitary representations of K_0. We define $\mathscr{E}(G) = \operatorname{Pol}(G)^*$ as the space of all linear maps from $\operatorname{Pol}(G)$ to $\mathbb{C}$. Note that $\mathscr{E}(G)$ can be identified with the space of finitely additive, complex measures on the space $\mathscr{F}(G)$ of compact open subsets of G.

When $K_0 < G$ is a compact open subgroup, we say that a map $\mu : G \to \mathscr{E}(G)$ is K_0-equivariant if

$$\mu(kxk') = \mu(x) \circ \operatorname{Ad} k^{-1} \quad \text{for all} \;\; k, k' \in K_0 \; .$$

Note that this implies that $\mu(x)$ is $\operatorname{Ad}(K_0 \cap xK_0x^{-1})$-invariant for all $x \in G$.

As in (3.4), we associate to every $x \in G$ and $\pi \in \operatorname{Irr}(K \cap xKx^{-1})$ the irreducible object $(\pi, x) \in \operatorname{Irr}(\mathscr{C})$ defined as the irreducible G-$L^\infty(G/K)$-$L^\infty(G/K)$-module $\mathscr{H}$ such that π is isomorphic with the representation of $K \cap xKx^{-1}$ on $1_{xK} \cdot \mathscr{H} \cdot 1_{eK}$. The formula

$$\varphi(\pi, x) = \dim(\pi)^{-1} \, \mu(x)(\chi_\pi) \tag{6.1}$$

then gives a bijection between the set of all functions $\varphi : \operatorname{Irr}(\mathscr{C}) \to \mathbb{C}$ and the set of all K-equivariant maps $\mu : G \to \mathscr{E}(G)$ with the property that $\mu(x)$ is supported on $K \cap xKx^{-1}$ for every $x \in G$.

Denote by $\mathscr{P} = \operatorname{Pol}(L^\infty(G) \rtimes_{\operatorname{Ad}} G)$ the dense $*$-subalgebra defined in (3.3). We always equip $\mathscr{P}$ with the operator space structure inherited from $\mathscr{P} \subset L^\infty(G) \rtimes_{\operatorname{Ad}} G$. As in Sect. 5, to every function $\varphi : \operatorname{Irr}(\mathscr{C}) \to \mathbb{C}$ is associated a linear map $\theta_\varphi : \mathscr{A} \to \mathscr{A}$ on the tube $*$-algebra $\mathscr{A}$ of $\mathscr{C}$. We now explain how to associate to any K_0-equivariant map $\mu : G \to \mathscr{E}(G)$ a linear map $\Psi_\mu : \mathscr{P} \to \mathscr{P}$. When φ and μ are related by (6.1) and $\Theta : \mathscr{A} \to \mathscr{P}$ is the $*$-anti-isomorphism of Theorem 3.1, it will turn out that $\Psi_\mu \circ \Theta$ equals $\Theta \circ \theta_\varphi$, so that in particular, $\|\theta_\varphi\|_{\mathrm{cb}}$ equals $\|\Psi_\mu\|_{\mathrm{cb}}$. We will further prove a criterion for Ψ_μ to be completely bounded and that will be the main tool to prove Theorem 6.1.

Denote $\Delta : L^\infty(G) \to L^\infty(G \times G) : \Delta(F)(g, h) = F(gh)$. For every $\mu \in \mathscr{E}(G)$, the linear map

$$\psi_\mu : \operatorname{Pol}(G) \to \operatorname{Pol}(G) : \psi_\mu(F) = (\mu \otimes \mathrm{id})\Delta(F)$$

is well defined. When $\mu : G \to \mathscr{E}(G)$ is K_0-equivariant with respect to the compact open subgroup $K_0 < G$, we define

$$\Psi_\mu : \mathscr{P} \to \mathscr{P} : \Psi_\mu(F \, u_x \, p_L) = \psi_{\mu(x)}(F) \, u_x \, p_L$$

for every $F \in \operatorname{Pol}(G)$, $x \in G$ and open subgroup $L < K_0$.

Lemma 6.2 *Denote by $\Theta : \mathscr{A} \to \mathscr{P}$ the $*$-anti-isomorphism constructed in Theorem 3.1 between the tube $*$-algebra $\mathscr{A}$ and $\mathscr{P} = \mathrm{Pol}(L^\infty(G) \rtimes_{\mathrm{Ad}} G)$. Let $\varphi : \mathrm{Irr}(\mathscr{C}) \to \mathbb{C}$ be any function and denote by $\mu : G \to \mathscr{E}(G)$ the associated K-equivariant map given by (6.1) with $\mu(x)$ supported in $K \cap xKx^{-1}$ for all $x \in G$. Then, $\Psi_\mu \circ \Theta = \Theta \circ \theta_\varphi$.*

Proof The result follows from a direct computation using (3.7). □

We prove the following technical result in exactly the same way as [10].

Lemma 6.3 *Let $K_0, K < G$ be compact open subgroups and $\mu : G \to \mathscr{E}(G)$ a K_0-equivariant map. Let $\kappa \geq 0$. Then the following conditions are equivalent.*

1. *Ψ_μ extends to a completely bounded map on $C_0(G) \rtimes^r_{\mathrm{Ad}} G$ with $\|\Psi_\mu\|_{\mathrm{cb}} \leq \kappa$.*
2. *Ψ_μ extends to a normal completely bounded map on $L^\infty(G) \rtimes_{\mathrm{Ad}} G$ satisfying $\|\Psi_\mu\|_{\mathrm{cb}} \leq \kappa$.*
3. *There exists a nondegenerate $*$-representation $\pi : C_0(G) \rtimes^f_{\mathrm{Ad}} G \to B(\mathscr{K})$ and bounded maps $V, W : G \to \mathscr{K}$ such that*
   - *$V(kxk') = \pi(k)V(x)$ and $W(kxk') = \pi(k)W(x)$ for all $x \in G$, $k \in K_0$ and $k' \in K$,*
   - *$\mu(zy^{-1})(F) = \langle \pi(F)\pi(zy^{-1})V(y), W(z)\rangle$ for all $F \in \mathrm{Pol}(G)$ and $y, z \in G$,*
   - *$\|V\|_\infty \|W\|_\infty \leq \kappa$.*

   In particular, every $\mu(x)$ is an actual complex measure on G, i.e. $\mu(x) \in C_0(G)^*$.

Proof 1 ⇒ 3. Denote $P = C_0(G) \rtimes^r_{\mathrm{Ad}} G$ and consider the (unique) completely bounded extension of Ψ_μ to P, which we still denote as Ψ_μ. Define the nondegenerate $*$-representation

$$\zeta : P \to B(L^2(G)) : \zeta(F) = F(e)1 \ \text{ and } \ \zeta(u_x) = \lambda_x$$

for all $F \in C_0(G)$, $x \in G$. Then $\zeta \circ \Psi_\mu : P \to B(L^2(G))$ has cb norm bounded by κ and satisfies

$$(\zeta \circ \Psi_\mu)(u_k S u_{k'}) = \lambda_k \, (\zeta \circ \Psi_\mu)(S) \, \lambda_{k'}$$

for all $S \in P$, $k, k' \in K_0$. By the Stinespring dilation theorem proved in [4, Theorem B.7], we can choose a nondegenerate $*$-representation $\pi : P \to B(\mathscr{K})$ and bounded operators $\mathscr{V}, \mathscr{W} : L^2(G) \to \mathscr{K}$ such that

- $(\zeta \circ \Psi_\mu)(S) = \mathscr{W}^* \pi(S) \mathscr{V}$ for all $S \in P$,
- $\mathscr{V}\lambda_k = \pi(k)\mathscr{V}$ and $\mathscr{W}\lambda_k = \pi(k)\mathscr{W}$ for all $k \in K_0$,
- $\|\mathscr{V}\| \, \|\mathscr{W}\| = \|\Psi_\mu\|_{\mathrm{cb}} \leq \kappa$.

We normalize the left Haar measure on G such that $\lambda(K) = 1$ and define the maps $V, W : G \to \mathscr{K}$ given by $V(y) = \mathscr{V}(1_{yK})$ and $W(z) = \mathscr{W}(1_{zK})$. By construction, 3 holds.

3 ⇒ 2. Write $P'' = L^\infty(G) \rtimes_{\mathrm{Ad}} G$. Denote by $\pi_r : P'' \to B(L^2(G \times G))$ the standard representation given by

$$(\pi_r(F)\xi)(g,h) = F(hgh^{-1})\xi(g,h) \quad \text{and} \quad (\pi_r(u_x)\xi)(g,h) = \xi(g,x^{-1}h)$$

for all $g,h,x \in G$, $F \in L^\infty(G)$.

For every nondegenerate $*$-representation $\pi : C_0(G) \rtimes^f_{\mathrm{Ad}} G \to B(\mathscr{K})$, there is a unique normal $*$-homomorphism $\tilde{\pi} : P'' \to B(\mathscr{K} \otimes L^2(G \times G))$ satisfying

$$\tilde{\pi}(F) = (\pi \otimes \pi_r)\Delta(F) \quad \text{and} \quad \tilde{\pi}(u_x) = \pi(x) \otimes \pi_r(x)$$

for all $F \in C_0(G)$, $x \in G$. Given V and W as in 3, we then define the bounded operators $\mathscr{V}, \mathscr{W} : L^2(G \times G) \to \mathscr{K} \otimes L^2(G \times G)$ by

$$(\mathscr{V}\xi)(g,h) = \xi(g,h)V(h) \quad \text{and} \quad (\mathscr{W}\xi)(g,h) = \xi(g,h)W(h)$$

for all $g,h \in G$. Note that $\|\mathscr{V}\| = \|V\|_\infty$ and $\|\mathscr{W}\| = \|W\|_\infty$. Since $\Psi_\mu(T) = \mathscr{W}^*\tilde{\pi}(T)\mathscr{V}$ for all $T \in \mathscr{P}$, it follows that 2 holds.

2 ⇒ 1 is trivial. □

We are now ready to prove Theorem 6.1. We follow closely the proof of [20, Theorem A].

Proof of Theorem 6.1 We define $\mathscr{Q}(G)$ as the set of all maps $\mu : G \to \mathscr{E}(G)$ satisfying the following properties:

- there exists a compact open subgroup $K_0 < G$ such that μ is K_0-equivariant,
- for every $x \in G$, we have that $\mu(x) \in C_0(G)^*$, $\mu(x)$ is compactly supported and $\mu(x)$ is absolutely continuous with respect to the Haar measure,
- $\|\Psi_\mu\|_{\mathrm{cb}} < \infty$.

Writing $\|\mu\|_{\mathrm{cb}} := \|\Psi_\mu\|_{\mathrm{cb}}$, we call a sequence $\mu_n \in \mathscr{Q}(G)$ a cbai (completely bounded approximate identity) if

- $\limsup_n \|\mu_n\|_{\mathrm{cb}} < \infty$,
- for every $F \in C_0(G)$, we have that $\mu_n(x)(F) \to F(e)$ uniformly on compact sets of $x \in G$,
- for every n, we have that μ_n has compact support (i.e. $\mu_n(x) = 0$ for all x outside a compact subset of G).

If a cbai exists, we define $\Gamma(G)$ as the smallest possible value of $\limsup_n \|\mu_n\|_{\mathrm{cb}}$, where (μ_n) runs over all cbai. Note that this smallest possible value is always attained by a cbai.

First assume that $\mathscr{C}$ is weakly amenable. By Proposition 5.1, we can take a sequence of finitely supported functions $\varphi_n : \mathrm{Irr}(\mathscr{C}) \to \mathbb{C}$ converging to 1 pointwise and satisfying $\limsup_n \|\theta_{\varphi_n}\|_{\mathrm{cb}} = \Lambda(\mathscr{C})$ where $\theta_{\varphi_n} : \mathscr{A} \to \mathscr{A}$ as before. Define the K-equivariant maps $\mu_n : G \to \mathscr{E}(G)$ associated with φ_n by (6.1).

For a fixed n and a fixed $x \in G$, there are only finitely many $\pi \in \mathrm{Irr}(K \cap xKx^{-1})$ such that $\varphi_n(\pi, x) \neq 0$. So, $\mu_n(x)$ is an actual complex measure on $K \cap xKx^{-1}$ that is absolutely continuous with respect to the Haar measure (and with the Radon-Nikodym derivative being in $\mathrm{Pol}(K \cap xKx^{-1})$). By Lemma 6.2, $\|\Psi_{\mu_n}\|_{\mathrm{cb}} = \|\theta_{\varphi_n}\|_{\mathrm{cb}} < \infty$. So, $\mu_n \in \mathscr{Q}(G)$ and the sequence (μ_n) is a cbai with $\limsup_n \|\mu_n\|_{\mathrm{cb}} \leq \Lambda(\mathscr{C})$. Thus, $\Gamma(G) \leq \Lambda(\mathscr{C})$. Write $\kappa = \Gamma(G)^{1/2}$.

For every map $\mu : G \to \mathscr{E}(G)$, we define

$$\overline{\mu} : G \to \mathscr{E}(G) : \overline{\mu}(x)(F) = \overline{(\mu(x^{-1}) \circ \mathrm{Ad}(x^{-1}))(\overline{F})} \; .$$

If μ is K_0-equivariant, also $\overline{\mu}$ is K_0-equivariant and $\Psi_{\overline{\mu}}(T) = (\Psi_\mu(T^*))^*$ for all $T \in \mathscr{P}$. So, $\|\overline{\mu}\|_{\mathrm{cb}} = \|\mu\|_{\mathrm{cb}}$. Also, if (μ_n) is a cbai, then $(\overline{\mu_n})$ is a cbai.

Since $\Gamma(G) = \kappa^2 < \infty$ and using Lemma 6.3, we can take a cbai (μ_n), a nondegenerate $*$-representation $\pi : C_0(G) \rtimes^f_{\mathrm{Ad}} G \to B(\mathscr{K})$ and bounded functions $V_n, W_n : G \to \mathscr{K}$ as in Lemma 6.3.3 with

$$\lim_n \|V_n\|_\infty = \kappa = \lim_n \|W_n\|_\infty \; .$$

Replacing μ_n by $(\mu_n + \overline{\mu_n})/2$, we may assume that $\mu_n = \overline{\mu_n}$ for all n. It then follows that both formulas

$$\begin{aligned} \mu_n(zy^{-1})(F) &= \langle \pi(F)\pi(zy^{-1})V_n(y), W_n(z)\rangle \quad \text{and} \\ \mu_n(zy^{-1})(F) &= \langle \pi(F)\pi(zy^{-1})W_n(y), V_n(z)\rangle \end{aligned}$$

hold for all $F \in C_0(G)$ and $y, z \in G$.

Put $\eta_n := \mu_n(e)$. We prove that $\|\eta_n \circ \mathrm{Ad}\, x - \eta_n\| \to 0$ uniformly on compact sets of $x \in G$. To prove this statement, fix an arbitrary compact subset $C \subset G$ and an arbitrary sequence $x_n \in C$. Define

$$\zeta_n : G \to \mathscr{E}(G) : \zeta_n(x) = \mu_n(x_n x) \circ \mathrm{Ad}\, x_n \; .$$

Since $\Psi_{\zeta_n}(T) = u_{x_n}^* \Psi_{\mu_n}(u_{x_n} T)$, it follows that (ζ_n) is a cbai. Also note that for all $y, z \in G$ and $F \in C_0(G)$, we have

$$\begin{aligned} \zeta_n(zy^{-1})(F) &= \langle \pi((\mathrm{Ad}\, x_n)(F))\, \pi(x_n z y^{-1})\, V_n(y), W_n(x_n z)\rangle \\ &= \langle \pi(F)\pi(zy^{-1})V_n(y), W_n'(z)\rangle \; , \end{aligned}$$

with $W_n'(z) = \pi(x_n)^* W_n(x_n z)$. Then also $(\mu_n + \zeta_n)/2$ is a cbai satisfying

$$\frac{1}{2}(\mu_n + \zeta_n)(zy^{-1})(F) = \langle \pi(F)\pi(zy^{-1})V_n(y), (W_n(z) + W_n'(z))/2\rangle$$

for all $y, z \in G$ and $F \in C_0(G)$. We conclude that

$$\kappa^2 \leq \liminf_n \|V_n\|_\infty \; \|(W_n + W_n')/2\|_\infty = \kappa \; \liminf_n \|(W_n + W_n')/2\|_\infty$$
$$\leq \kappa \; \limsup_n \|(W_n + W_n')/2\|_\infty \leq \kappa \; \frac{1}{2} \limsup_n \big(\|W_n\|_\infty + \|W_n'\|_\infty\big) = \kappa^2 \; .$$

Therefore, $\lim_n \|(W_n + W_n')/2\|_\infty = \kappa$. So, we can choose $z_n \in G$ such that

$$\lim_n \|(W_n(z_n) + W_n'(z_n))/2\| = \kappa \; .$$

Since also $\limsup_n \|W_n(z_n)\| \leq \kappa$ and $\limsup_n \|W_n'(z_n)\| \leq \kappa$, the parallelogram law implies that $\lim_n \|W_n(z_n) - W_n'(z_n)\| = 0$.

Since for all $F \in C_0(G)$,

$$\zeta_n(e)(F) = \zeta_n(z_n z_n^{-1})(F) = \langle \pi(F) V_n(z_n), W_n'(z_n)\rangle \quad \text{and}$$
$$\mu_n(e)(F) = \mu_n(z_n z_n^{-1})(F) = \langle \pi(F) V_n(z_n), W_n(z_n)\rangle \; ,$$

it follows that $\lim_n \|\zeta_n(e) - \mu_n(e)\| = 0$. This means that

$$\lim_n \|\mu_n(x_n) \circ \operatorname{Ad} x_n - \mu_n(e)\| = 0 \; .$$

Since the compact set $C \subset G$ and the sequence $x_n \in C$ were arbitrary, we have proved that $\lim_n \|\mu_n(x) - \mu_n(e) \circ \operatorname{Ad} x^{-1}\| = 0$ uniformly on compact sets of $x \in G$.

Reasoning in a similar way with $\zeta_n : G \to \mathscr{E}(G) : \zeta_n(x) = \mu_n(x x_n^{-1})$, which satisfies

$$\zeta_n(zy^{-1})(F) = \langle \pi(F)\pi(zy^{-1}) V_n'(y), W_n(z)\rangle$$

with $V_n'(y) = \pi(x_n)^* V_n(x_n y)$, we also find that $\lim_n \|\mu_n(x) - \mu_n(e)\| = 0$ uniformly on compact sets of $x \in G$. Both statements together imply that $\|\eta_n \circ \operatorname{Ad} x - \eta_n\| \to 0$ uniformly on compact sets of $x \in G$.

We next claim that for every $H \in \operatorname{Pol}(G)$ with $H(e) = 1$ and $\|H\|_\infty = 1$, we have that $\lim_n \|\eta_n \cdot H - \eta_n\| = 0$. To prove this claim, define

$$\zeta_n : G \to \mathscr{E}(G) : \zeta_n(x)(F) = \mu_n(x)(HF) \; .$$

Since $\zeta_n(zy^{-1})(F) = \langle \pi(F)\pi(zy^{-1}) V_n(y), W_n'(z)\rangle$ with $W_n'(z) = \pi(H)^* W_n(z)$ and because the function $H \in \operatorname{Pol}(G)$ is both left and right K_0-invariant for a small enough compact open subgroup $K_0 < G$, it follows from Lemma 6.3 that

$$\|\zeta_n\|_{\mathrm{cb}} \leq \|V_n\|_\infty \, \|W_n'\|_\infty \leq \|V_n\|_\infty \, \|W_n\|_\infty = \|\mu_n\|_{\mathrm{cb}} \; .$$

So again, (ζ_n) and $(\mu_n + \zeta_n)/2$ are cbai. The same reasoning as above gives us a sequence $z_n \in G$ with $\lim_n \|W_n(z_n) - W'_n(z_n)\| = 0$, which allows us to conclude that $\lim_n \|\mu_n(e) - \zeta_n(e)\| = 0$, thus proving the claim.

Altogether, we have proved that $\eta_n \in C_0(G)^*$ is a sequence of complex measures that are absolutely continuous with respect to the Haar measure and that satisfy

- $\|\eta_n - \eta_n \circ \mathrm{Ad}\, x\| \to 0$ uniformly on compact sets of $x \in G$,
- $\|\eta_n \cdot 1_L - \eta_n\| \to 0$ for every compact open subset $L \subset G$ with $e \in L$,
- $\eta_n(F) \to F(e)$ for every $F \in C_0(G)$.

In particular, $\liminf_n \|\eta_n\| \geq 1$. But then $\omega_n := \|\eta_n\|^{-1} |\eta_n|$ is a sequence of probability measures on G that are absolutely continuous with respect to the Haar measure and satisfy $\omega_n \to \delta_e$ weakly* and $\|\omega_n \circ \mathrm{Ad}\, x - \omega_n\| \to 0$ uniformly on compact sets of $x \in G$.

By Lemma 6.3, the maps Ψ_{μ_n} extend to normal cb maps on $L^\infty(G) \rtimes_{\mathrm{Ad}} G$. Restricting to $L(G)$, we obtain the compactly supported Herz-Schur multipliers

$$L(G) \to L(G) : u_x \mapsto \gamma_n(x) u_x \quad \text{for all} \;\; x \in G \; ,$$

where $\gamma_n : G \to \mathbb{C}$ is the compactly supported, locally constant function given by $\gamma_n(x) = \mu_n(x)(1)$. So, G is weakly amenable and

$$\Lambda(G) \leq \limsup_n \|\Psi_{\mu_n}|_{L(G)}\|_{\mathrm{cb}} \leq \limsup_n \|\Psi_{\mu_n}\| \leq \Lambda(\mathscr{C}) \; .$$

Conversely, assume that G is weakly amenable and that there exists a sequence of probability measures $\omega_n \in C_0(G)^*$ that are absolutely continuous with respect to the Haar measure and such that $\omega_n \to \delta_e$ weakly* and $\|\omega_n \circ \mathrm{Ad}\, x - \omega_n\| \to 0$ uniformly on compact sets of $x \in G$.

Since G is weakly amenable, we can take a sequence of K-biinvariant Herz-Schur multipliers $\zeta_n : G \to \mathbb{C}$ having compact support, converging to 1 uniformly on compacta and satisfying $\limsup_n \|\zeta_n\|_{\mathrm{cb}} = \Lambda(G)$.

Denote by $\mathrm{Pol}(G)^+$ the set of positive, locally constant, compactly supported functions on G. Denote by $h \in C_0(G)^*$ the Haar measure on the compact open subgroup $K < G$. Approximating ω_n, we may assume that $\omega_n = h \cdot \xi_n^2$, where ξ_n is a sequence of $\mathrm{Ad}\, K$-invariant functions in $\mathrm{Pol}(K)^+$. Define the representation $\pi : C_0(G) \rtimes^f_{\mathrm{Ad}} G \to B(L^2(G))$ given by

$$(\pi(F)\xi)(g) = F(g)\xi(g) \quad \text{and} \quad (\pi(x)\xi)(g) = \Delta(x)^{1/2}\, \xi(x^{-1}gx)$$

for all $F \in C_0(G)$, $\xi \in L^2(G)$ and $x, g \in G$. We then define the K-equivariant map

$$\mu_n : G \to C_0(G)^* : \mu_n(x)(F) = \zeta_n(x) \, \langle \pi(F)\pi(x)\xi_n, \xi_n \rangle \; .$$

Since ξ_n is an $\mathrm{Ad}\, K$-invariant element of $\mathrm{Pol}(K)$ and $\pi(x)\xi_n$ is an $\mathrm{Ad}(xKx^{-1})$-invariant element of $\mathrm{Pol}(xKx^{-1})$, we get that $\mu_n(x)$ is an $\mathrm{Ad}(K \cap xKx^{-1})$-invariant

complex measure supported on $K \cap xKx^{-1}$ and having a density in $\mathrm{Pol}(K \cap xKx^{-1})$ with respect to the Haar measure. Since moreover ζ_n is compactly supported, it follows that the functions $\varphi_n : \mathrm{Irr}(\mathscr{C}) \to \mathbb{C}$ associated with μ_n through (6.1) are finitely supported.

Since $\|\omega_n \circ \mathrm{Ad}\, x - \omega_n\| \to 0$ for every $x \in G$, we have that $\|\pi(x)\xi_n - \xi_n\| \to 0$ for every $x \in G$. Since $\omega_n \to \delta_e$ weakly*, we have that $\langle \pi(F)\xi_n, \xi_n\rangle \to F(e)$ for every $F \in C_0(G)$. Both together imply that $\varphi_n \to 1$ pointwise.

To conclude the proof of the theorem, by Lemma 6.2, it suffices to prove that $\limsup_n \|\mu_n\|_{\mathrm{cb}} \leq \Lambda(G)$.

Since ζ_n is a K-biinvariant Herz-Schur multiplier on G, we can choose a Hilbert space $\mathscr{K}$ and K-biinvariant functions $V_n, W_n : G \to \mathscr{K}$ such that

$$\|V_n\|_\infty \, \|W_n\|_\infty = \|\zeta_n\|_{\mathrm{cb}} \quad \text{and} \quad \zeta_n(zy^{-1}) = \langle V_n(y), W_n(z)\rangle \tag{6.2}$$

for all $y, z \in G$. We equip $L^2(G) \otimes \mathscr{K}$ with the $*$-representation of $C_0(G) \rtimes^f_{\mathrm{Ad}} G$ given by $\pi(\,\cdot\,) \otimes 1$. We define the bounded maps

$$\mathscr{V}_n : G \to L^2(G) \otimes \mathscr{K} : \mathscr{V}_n(y) = \xi_n \otimes V_n(y) \quad \text{and}$$
$$\mathscr{W}_n : G \to L^2(G) \otimes \mathscr{K} : \mathscr{W}_n(y) = \xi_n \otimes W_n(y) \; .$$

One checks that

$$\mu_n(zy^{-1})(F) = \langle (\pi(F)\pi(zy^{-1}) \otimes 1)\mathscr{V}_n(y), \mathscr{W}_n(z)\rangle$$

for all y, z and that all other conditions in Lemma 6.3.3 are satisfied, with $\|\mathscr{V}_n\|_\infty = \|V_n\|_\infty$ and $\|\mathscr{W}_n\|_\infty = \|W_n\|_\infty$. So, we conclude that

$$\limsup_n \|\mu_n\|_{\mathrm{cb}} \leq \limsup_n \|\zeta_n\|_{\mathrm{cb}} = \Lambda(G)$$

and this ends the proof of the theorem. □

Example 6.4 Taking G as in Example 4.4, the category $\mathscr{C}$ is weakly amenable with $\Lambda(\mathscr{C}) = 1$. Indeed, G is weakly amenable with $\Lambda(G) = 1$ and the probability measures μ_n constructed in Example 4.4 are absolutely continuous with respect to the Haar measure, so that the result follows from Theorem 6.1.

Taking $G = \mathrm{SL}(2, F)$ as in Proposition 4.2, we get that $\mathscr{C}$ is not weakly amenable, although G is weakly amenable with $\Lambda(G) = 1$.

Acknowledgements Y.A. was supported by the Research Fellow of the Japan Society for the Promotion of Science and the Program for Leading Graduate Schools, MEXT, Japan.

S.V. was supported in part by European Research Council Consolidator Grant 614195, and by long term structural funding—Methusalem grant of the Flemish Government.

References

1. Y. Arano, Unitary spherical representations of Drinfeld doubles. *J. Reine Angew. Math.*, to appear. arXiv:1410.6238. doi:10.1515/crelle-2015-0079
2. B. Bekka, P. de la Harpe and A. Valette, Kazhdan's property (T). Cambridge University Press, Cambridge, 2008.
3. A. Brothier, Weak amenability for subfactors. *Internat. J. Math.* **26** (2015), art. id. 1550048.
4. N.P. Brown and N. Ozawa, C*-algebras and finite-dimensional approximations. *Graduate Studies in Mathematics* **88**. American Mathematical Society, Providence, 2008.
5. R.D. Burstein, Automorphisms of the bipartite graph planar algebra. *J. Funct. Anal.* **259** (2010), 2384–2403.
6. K. De Commer, A. Freslon and M. Yamashita, CCAP for universal discrete quantum groups. *Comm. Math. Phys.* **331** (2014), 677–701.
7. S. Deprez and S. Vaes, A classification of all finite index subfactors for a class of group-measure space II$_1$ factors. *J. Noncommut. Geom.* **5** (2011), 523–545.
8. Y. de Cornulier, Finitely presentable, non-Hopfian groups with Kazhdan's property (T) and infinite outer automorphism group. *Proc. Amer. Math. Soc.* **135** (2007), 951–959.
9. S. Ghosh and C. Jones, Annular representation theory for rigid C*-tensor categories. *J. Funct. Anal.* **270** (2016), 1537–1584.
10. P. Jolissaint, A characterization of completely bounded multipliers of Fourier algebras. *Colloq. Math.* **63** (1992), 311–313.
11. V.F.R Jones, The planar algebra of a bipartite graph. In *Knots in Hellas '98*, World Scientific, 1999, pp. 94–117.
12. V.F.R. Jones, Planar algebras, I. arXiv:math.QA/9909027
13. R. Longo and J.E. Roberts, A theory of dimension. *K-Theory* **11** (1997), 103–159.
14. S. Morrison, E. Peters and N. Snyder, Skein theory for the D_{2n} planar algebras. *J. Pure Appl. Algebra* **214** (2010), 117–139.
15. M. Müger, From subfactors to categories and topology, I. Frobenius algebras in and Morita equivalence of tensor categories. *J. Pure Appl. Algebra* **180** (2003), 81–157.
16. S. Neshveyev and L. Tuset, Compact quantum groups and their representation categories. *Cours Spécialisés* **20**. Société Mathématique de France, Paris, 2013.
17. S. Neshveyev and M. Yamashita, Drinfeld center and representation theory for monoidal categories. *Commun. Math. Phys.*, to appear. arXiv:1501.07390. doi:10.1007/s00220-016-2642-7
18. S. Neshveyev and M. Yamashita, A few remarks on the tube algebra of a monoidal category. *Preprint.* arXiv:1511.06332
19. A. Ocneanu, Chirality for operator algebras. In *Subfactors (Kyuzeso, 1993)*, World Sci. Publ., River Edge, 1994, pp. 39–63.
20. N. Ozawa, Examples of groups which are not weakly amenable. *Kyoto J. Math.* **52** (2012), 333–344.
21. S. Popa, Symmetric enveloping algebras, amenability and AFD properties for subfactors. *Math. Res. Lett.* **1** (1994), 409–425.
22. S. Popa, An axiomatization of the lattice of higher relative commutants of a subfactor. *Invent. Math.* **120** (1995), 427–445.
23. S. Popa, Some properties of the symmetric enveloping algebra of a subfactor, with applications to amenability and property T. *Doc. Math.* **4** (1999), 665–744.
24. S. Popa, On a class of type II$_1$ factors with Betti numbers invariants. *Ann. of Math.* **163** (2006), 809–899.
25. S. Popa, D. Shlyakhtenko and S. Vaes, Cohomology and L^2-Betti numbers for subfactors and quasi-regular inclusions. *Preprint.* arXiv:1511.07329
26. S. Popa and S. Vaes, Representation theory for subfactors, λ-lattices and C*-tensor categories, *Commun. Math. Phys.* **340** (2015), 1239–1280.

27. S. Vaes, Strictly outer actions of groups and quantum groups. *J. Reine Angew. Math.* **578** (2005), 147–184.
28. A. Wassermann, Ergodic actions of compact groups on operator algebras, I. *Ann. of Math.* **130** (1989), 273–319.
29. Y. Zhu, Hecke algebras and representation ring of Hopf algebras. In *First International Congress of Chinese Mathematicians (Beijing, 1998)*, AMS/IP Stud. Adv. Math. **20**, Amer. Math. Soc., Providence, 2001, pp. 219–227.

Decomposable Approximations Revisited

Nathanial P. Brown, José R. Carrión, and Stuart White

Abstract Nuclear C*-algebras enjoy a number of approximation properties, most famously the completely positive approximation property. This was recently sharpened to arrange for the incoming maps to be sums of order-zero maps. We show that, in addition, the outgoing maps can be chosen to be asymptotically order-zero. Further these maps can be chosen to be asymptotically multiplicative if and only if the C*-algebra and all its traces are quasidiagonal.

1 Introduction

Approximation properties are ubiquitous in operator algebras, characterizing many key notions and providing essential tools. In particular, and central to this note, a foundational result of Choi-Effros [10] and Kirchberg [18] describes nuclearity of a C*-algebra in terms of completely positive approximations. Precisely, A is nuclear if and only if there exist finite dimensional algebras (F_i) and completely positive contractive (c.p.c.) maps

$$A \xrightarrow{\psi_i} F_i \xrightarrow{\phi_i} A \tag{1.1}$$

N.P. Brown
Department of Mathematics, The Pennsylvania State University, University Park, State College, PA 16802, USA
e-mail: nbrown@math.psu.edu

J.R. Carrión
Department of Mathematics, Texas Christian University, Fort Worth, TX 76129, USA
e-mail: j.carrion@tcu.edu

S. White (✉)
School of Mathematics and Statistics, University of Glasgow, Glasgow G12 8QW, Scotland, UK
Mathematisches Institut der WWU Münster, Einsteinstraße 62, 48149 Münster, Germany
e-mail: stuart.white@glasgow.ac.uk

T.M. Carlsen et al. (eds.), *Operator Algebras and Applications*, Abel Symposia 12,
DOI 10.1007/978-3-319-39286-8_2

that approximate the identity in the point-norm topology, i.e.

$$\lim_i \|\phi_i(\psi_i(x)) - x\| = 0, \quad x \in A. \tag{1.2}$$

Some 30 years later, via Connes' celebrated work on injective von Neumann algebras [9], this approximation property was shown to imply a stronger version of itself: one can always take every ϕ_i to be a convex combination of contractive order-zero maps [16]. This has proved crucial to applications to near inclusions (for example, [16, Theorem 2.3]). In this note we observe a further improvement: every ψ_i can be taken to be asymptotically order zero, meaning that if $a, b \in A$ are self-adjoint and $ab = 0$, then

$$\lim_i \|\psi_i(a)\psi_i(b)\| = 0. \tag{1.3}$$

It was known that (1.3) could be arranged under the stronger hypothesis of finite nuclear dimension [29, Proposition 3.2] and this proved vital to various applications (cf. [7, 21, 26, 27]).

Our proof follows the strategy in [16] by obtaining suitable factorizations of the canonical inclusion $A \hookrightarrow A^{**}$ with respect to the weak* topology; then adjusting these to take values in A; and finally applying a Hahn-Banach argument to get asymptotic factorizations in the point-norm topology. To do this in general, however, we require some quasidiagonal ideas. Indeed, the main technical hurdle is showing that if A is quasidiagonal and all traces on A are quasidiagonal in the sense of [6], then one can take every ψ_i to be asymptotically multiplicative (see Theorem 2.2), while retaining the decomposition of ϕ_i as a convex combination of contractive order zero maps. This should be compared with Blackadar and Kirchberg's characterization of nuclear quasidiagonal C*-algebras in [3] as those with approximations (1.1) and (1.2) with ψ_i asymptotically multiplicative.

Since all traces on nuclear quasidiagonal C*-algebras in the UCT class are quasidiagonal [24], our result improves the Blackadar-Kirchberg characterization in this case. Cones over nuclear C*-algebras are quasidiagonal [25] and satisfy the UCT, so all their traces are quasidiagonal (though we show how Gabe's work [14] gives a simpler proof of this fact in Proposition 3.2). Thus we obtain our main theorem for general nuclear A by taking an order-zero splitting $A \to CA$, applying the improved approximation maps on CA, then using the quotient map $CA \to A$ to get back to A (see the proof of Theorem 3.1 for details).

2 Quasidiagonal Traces

In this note, a *trace* on a C^*-algebra means a tracial state. Write $T(A)$ for the collection of all traces on A. Various approximation properties for traces were studied in [6]; of particular relevance here is the notion of quasidiagonality for traces.

Definition 2.1 A trace τ on a C^*-algebra A is *quasidiagonal* if there exist finite dimensional algebras F_i, tracial states tr_i on F_i and c.p.c. maps $\theta_i : A \to F_i$ such that $\mathrm{tr}_i \circ \theta_i \to \tau$ in the weak* topology and

$$\lim_i \|\theta_i(ab) - \theta_i(a)\theta_i(b)\| = 0 \tag{2.1}$$

for all $a, b \in A$. Write $T_{\mathrm{qd}}(A)$ for the set of quasidiagonal traces of A.

When A is unital the maps θ_i can be taken to be unital and completely positive (u.c.p.). Theorem 3.1.6 of [6] lists several other characterizations of amenable traces.

The main technical result of this note is the following.

Theorem 2.2 *Let A be a separable and nuclear C^*-algebra. Then A is quasidiagonal and $T(A) = T_{\mathrm{qd}}(A)$ if and only if there exist a sequence of finite-dimensional C^*-algebras (F_n) and c.p.c. maps*

$$A \xrightarrow{\psi_n} F_n \xrightarrow{\phi_n} A \tag{2.2}$$

such that

1. $\|(\phi_n \circ \psi_n)(a) - a\| \to 0$ *for all $a \in A$;*
2. *every ϕ_n is a convex combination of finitely many contractive order zero maps; and*
3. $\|\psi_n(ab) - \psi_n(a)\psi_n(b)\| \to 0$ *for all $a, b \in A$.*

Only one implication of this theorem requires much work. Indeed, if A has approximations with properties 1–3, then A is quasidiagonal (this is an easy implication in [3, Theorem 5.2.2]; the maps ψ_i are approximately multiplicative by 3, and 1 ensures that they are approximately isometric). It is equally routine to check that all traces are quasidiagonal. Indeed, since a trace composed with an order-zero map is a trace by [28, Corollary 4.4], and each ϕ_n is a convex combination of order zero maps, given a trace $\tau_A \in T(A)$, it follows that $\tau_A \circ \phi_n$ defines a trace on F_n. Then condition 1 ensures that $(\tau_A \circ \phi_n) \circ \psi_n \to \tau_A$ weak*.

In order to prove the reverse implication it will suffice to prove a σ-weak version for the canonical inclusion $\iota : A \hookrightarrow A^{**}$. Namely, we prove the following proposition in the remainder of this section.

Proposition 2.3 *Let A be a separable nuclear, and quasidiagonal C^*-algebra with $T(A) = T_{\mathrm{qd}}(A)$. Then there are nets of finite-dimensional C^*-algebras (F_i) and of c.p.c. maps*

$$A \xrightarrow{\psi_i} F_i \xrightarrow{\phi_i} A^{**} \tag{2.3}$$

such that

1. $(\phi_i \circ \psi_i)(a) \to \iota(a)$ *in the* σ*-weak topology for every* $a \in A$*;*
2. ϕ_i *is an order zero map;*
3. $\|\psi_i(ab) - \psi_i(a)\psi_i(b)\| \to 0$ *for every* $a, b \in A$.

With this proposition in hand we prove Theorem 2.2 by following the same steps used to prove [16, Theorem 1.4] from the preparatory lemma [16, Lemma 1.3]. Indeed, using the notation of Proposition 2.3, first apply Lemma 1.1 of [16] to see that for every i there is a net of contractive order zero maps $(\phi_{i,\lambda}\colon F_i \to A)_\lambda$ such that $\phi_{i,\lambda}(x)$ converges σ-weakly to $\phi_i(x)$ for every $x \in F_i$. We may therefore assume that the image of ϕ_i is contained in A for every i. The argument now ends with a familiar Hahn-Banach argument, similar to the one used to prove the completely positive approximation property of a C^*-algebra from the assumption that its enveloping von Neumann algebra is semidiscrete (see [5, Proposition 2.3.8]). Briefly, given a finite subset $\mathscr{F}$ of A and $\epsilon > 0$, let $K_0 \subset \mathscr{B}(A)$ be the collection of all c.p.c maps $\theta\colon A \to A$ which factorize as $A \overset{\psi}{\to} F \overset{\phi}{\to} A$, where ψ is a c.p.c. map with $\|\psi(ab) - \psi(a)\psi(b)\| \leq \epsilon$ for all $a, b \in \mathscr{F}$, and ϕ is a contractive order zero map. Since the identity map on A lies in the point-weak closure of K_0, it lies in the point norm closure of the convex hull of K_0. As a convex combination of maps in K_0 can be factorized in the form $A \overset{\psi}{\to} F \overset{\phi}{\to} A$, where ψ is a c.p.c. map with $\|\psi(ab) - \psi(a)\psi(b)\| \leq \epsilon$ for all $a, b \in \mathscr{F}$ and ϕ a convex combination of contractive order zero maps, we can find such ψ and ϕ additionally satisfying $\|\phi(\psi(a)) - a\| < \epsilon$ for $a \in \mathscr{F}$. Theorem 2.2 follows by using a countable dense subset of A to produce the required sequence of maps.

The proof of Proposition 2.3 requires some lemmas and will first be carried out in the case when A is unital. We will split A^{**} into two pieces, the finite and properly infinite summands, and then handle each piece separately.[1] The properly infinite case is handled by a combination of Blackadar and Kirchberg's characterization of NF-algebras in [3] and Haagerup's very short proof that semidiscreteness implies hyperfiniteness for properly infinite von Neumann algebras [15, Section 2].

Recall that if ρ is a normal state on a von Neumann algebra M, the seminorm $\|\cdot\|_\rho^\sharp$ is given by

$$\|x\|_\rho^\sharp = \rho\left(\frac{xx^* + x^*x}{2}\right)^{1/2}, \quad x \in M. \tag{2.4}$$

It is a standard fact (see e.g. [4, III.2.2.19]) that if $\{\rho_i\}$ is a separating family of normal states on M, then the topology generated by $\{\|\cdot\|_{\rho_i}^\sharp\}$ agrees with the σ-strong* topology on any bounded subset of M. This will be used in both of the following lemmas.

[1]Recall that a von Neumann algebra is finite if it admits a separating family of tracial states, and properly infinite if it has no finite summand.

Lemma 2.4 *Let A be a unital, quasidiagonal and nuclear* C*-*algebra. Let π_∞ : $A \to M$ be the properly infinite summand of the universal representation of A. Then there are nets of finite-dimensional C^*-algebras F_i and nets of c.p.c. maps*

$$A \xrightarrow{\psi_i} F_i \xrightarrow{\phi_i} M \tag{2.5}$$

such that

1. $(\phi_i \circ \psi_i)(a) \to \pi_\infty(a)$ *in the σ-strong* topology (and hence also in the σ-weak topology) for every $a \in A$;*
2. ϕ_i *is a *-homomorphism for every i; and*
3. $\|\psi_i(ab) - \psi_i(a)\psi_i(b)\| \to 0$ *for every $a, b \in A$.*

Proof Fix $\epsilon > 0$, a finite subset $\mathscr{F}$ of unitaries in A, and finitely many normal states $\rho_1, \ldots, \rho_m$ on M. We will produce a factorization

$$A \xrightarrow{\psi} F \xrightarrow{\phi} M \tag{2.6}$$

where F is a matrix algebra, ϕ is a *-homomorphism and ψ is a u.c.p. map, such that

$$\|\phi(\psi(u)) - u\|_{\rho_i}^\sharp < 2\epsilon^{\frac{1}{2}} \tag{2.7}$$

and

$$\|\psi(uv) - \psi(u)\psi(v)\| < \epsilon, \tag{2.8}$$

for all $u, v \in \mathscr{F}$ and $i = 1, \ldots, m$. In this way we obtain the desired net indexed by finite subsets of unitaries, finite subsets of normal states and tolerances ϵ. By working with $\rho = \frac{1}{m}\sum_{i=1}^m \rho_i$, and replacing ϵ by ϵ/m, it suffices to obtain the single estimate

$$\|\phi(\psi(u)) - u\|_\rho^\sharp < 2\epsilon^{\frac{1}{2}}, \quad u \in \mathscr{F}, \tag{2.9}$$

in place of (2.7).

Since A is nuclear and quasidiagonal, it is NF by [3, Theorem 5.2.2] and so, by part (vi) of that theorem, there exists a matrix algebra F and u.c.p. maps

$$A \xrightarrow{\psi} F \xrightarrow{\theta} A \tag{2.10}$$

such that

$$\|(\theta \circ \psi)(u) - u\| < \epsilon \tag{2.11}$$

and

$$\|\psi(uv) - \psi(u)\psi(v)\| < \epsilon, \tag{2.12}$$

for all $u, v \in \mathscr{F}$. The estimate in (2.11) gives

$$\|\pi_\infty(\theta(\psi(u)) - u)\|_\rho^\sharp < \epsilon, \tag{2.13}$$

for all $u \in \mathscr{F}$.

We now follow the proof of [15, Theorem 2.2]. As M is properly infinite, we can fix a unital embedding $\iota : F \to M$. Then by [15, Proposition 2.1] there exists an isometry $v \in M$ such that $\theta(x) = v^*\iota(x)v$ for all $x \in F$. If v is a unitary (which is impossible, in general), then we're done because $\mathrm{Ad}(v) \circ \iota$ is the desired $*$-homomorphism. Since the σ-strong closure of unitaries in any von Neumann algebra is the set of all isometries (cf. [23, Lemma XVI.1.1]), the remainder of the proof (which follows the estimates on page 167 of [15]) amounts to approximating v with a suitable unitary.

We may assume that M is concretely represented on some Hilbert space $\mathscr{H}$ so that ρ is a vector state, given by a unit vector $\xi \in \mathscr{H}$. Using the identity $\|x\xi\|^2 + \|x^*\xi\|^2 = 2(\|x\|_\rho^\sharp)^2$, which is valid for all $x \in M$, and Eq. (2.13) we have

$$\|(v^*\iota(\psi(u))v - \pi_\infty(u))\xi\| < 2^{\frac{1}{2}}\epsilon \tag{2.14}$$

and

$$\|(v^*\iota(\psi(u)^*)v - \pi_\infty(u^*))\xi\| < 2^{\frac{1}{2}}\epsilon. \tag{2.15}$$

This implies

$$\Re\langle \iota(\psi(u))v\xi, v\pi_\infty(u)\xi\rangle > 1 - 2^{\frac{1}{2}}\epsilon \tag{2.16}$$

and

$$\Re\langle \iota(\psi(u)^*)v\xi, v\pi_\infty(u^*)\xi\rangle > 1 - 2^{\frac{1}{2}}\epsilon. \tag{2.17}$$

Now choose a unitary $w \in M$ such that, for all $u \in \mathscr{F}$,

$$\Re\langle \iota(\psi(u))w\xi, w\pi_\infty(u)\xi\rangle > 1 - 2\epsilon \tag{2.18}$$

and

$$\Re\langle \iota(\psi(u)^*)w\xi, w\pi_\infty(u^*)\xi\rangle > 1 - 2\epsilon. \tag{2.19}$$

Then, since $\|\iota(\psi(u))w\xi\| \leq 1$ and $\|\iota(\psi(u^*))w\xi\| \leq 1$, we have

$$\|\iota(\psi(u))w\xi - w\pi_\infty(u)\xi\|^2 \leq 2 - 2\Re\langle\iota(\psi(u))w\xi, w\pi_\infty(u)\xi\rangle < 4\epsilon \tag{2.20}$$

and

$$\|\iota(\psi(u^*))w\xi - w\pi_\infty(u^*)\xi\|^2 \leq 2 - 2\Re\langle\iota(\psi(u^*))w\xi, w\pi_\infty(u^*)\xi\rangle < 4\epsilon, \tag{2.21}$$

for all $u \in \mathscr{F}$. Then $\phi = \mathrm{Ad}(w^*) \circ \iota : F \to M$ is a *-homomorphism with

$$\|\phi(\psi(u)) - \pi_\infty(u)\|_\rho^\sharp < (4\epsilon)^{\frac{1}{2}}, \quad u \in \mathscr{F}, \tag{2.22}$$

as required. □

Next we deal with the finite part of A^{**}. We need the following standard uniqueness fact. Let A be a separable nuclear C^*-algebra, and N a finite von Neumann algebra. Then it is well known, though most often stated when N is a factor (see [17] and [1] which give converse statements), or when N has separable predual (see [11, Theorem 5]) that two *-homomorphisms $\phi_1, \phi_2 : A \to N$ are σ-strong* approximately unitarily equivalent in that there is a net of unitaries u_i such that $u_i\phi_1(a)u_i^* \to \phi_2(a)$ in the σ-strong* topology for all $a \in A$ if and only if $\tau \circ \phi_1 = \tau \circ \phi_2$ for all normal traces τ on N. Indeed, ϕ_1 and ϕ_2 extend to normal representations $\phi_1^{**}, \phi_2^{**} : A^{**} \to N$ that agree on traces. Since A^{**} is injective, it is hyperfinite,[2] so there is an increasing net of finite dimensional subalgebras (F_λ) that is σ-strong* dense in A^{**}. For each λ, the condition that $\tau \circ \phi_1^{**}|_{F_\lambda} = \tau \circ \phi_2|_{F_\lambda}^{**}$ for all normal traces τ on N gives a unitary u_λ with $\mathrm{Ad}(u_\lambda) \circ \phi_1^{**}|_{F_\lambda} = \phi_2^{**}|_{F_\lambda}$. The net of unitaries (u_λ) witnesses the σ-strong* approximate unitary equivalence of ϕ_1^{**} and ϕ_2^{**} and hence also of ϕ_1 and ϕ_2.

Lemma 2.5 *Let A be a separable, unital and nuclear C^*-algebra and assume $T(A) = T_{\mathrm{qd}}(A)$. Let $\pi_{\mathrm{fin}}: A \to M$ be the finite summand of the universal representation of A. Then there are nets of finite dimensional C^*-algebras F_i and of c.p.c. maps*

$$A \xrightarrow{\psi_i} F_i \xrightarrow{\phi_i} M \tag{2.23}$$

such that

1. *$(\phi_i \circ \psi_i)(a) \to \pi_{\mathrm{fin}}(a)$ in the σ-strong* topology (and therefore also in the σ-weak topology) for every $a \in A$;*

[2] See [12] for the extension of Connes' theorem to the non-separable predual case used here.

2. *ϕ_i is a *-homomorphism for every n; and*
3. $\|\psi_i(ab) - \psi_i(a)\psi_i(b)\| \to 0$ *for every $a, b \in A$.*

Proof Recall that M has a separating family of normal tracial states. As pointed out in the remarks preceding Lemma 2.4, on any bounded subset of M the σ-strong* topology agrees with the topology generated by the family of seminorms $\{\|\cdot\|_{2,\tau}\}$ (where τ runs through all normal tracial states of M). As in the proof of Lemma 2.4, the required nets of finite dimensional C^*-algebras and c.p.c. maps will ultimately be indexed by finite subsets $\mathscr{F}$ of A, positive numbers ϵ, and finite subsets $\{\tau_1, \ldots, \tau_m\}$ of normal tracial states of M. Moreover, the same argument found in the proof of Lemma 2.4 shows that it suffices to consider a single normal trace τ (by considering $\tau = \frac{1}{m}\sum_{i=1}^m \tau_i$), which we fix for the remainder of the proof.

Write N for $\pi_\tau(A)''$. We claim it is enough to obtain finite dimensional algebras F_i and maps $\psi_i: A \to F_i$ and $\phi_i: F_i \to N$ (as opposed to $\phi_i: F_i \to M$) satisfying (2), (3), and

$$\|(\phi_i \circ \psi_i)(a) - \pi_\tau(a)\|_{2,\tau} \to 0. \tag{2.24}$$

For this, first note that $J = \{x \in M : \tau(x^*x) = 0\}$ is a (closed, two-sided) ideal of M, and therefore of the form Mp for some central projection $p \in M$. Using the fact that τ is a faithful trace on both N and $M(1-p)$, we get that $N \cong M(1-p)$ (extending the identity on $A/J \cap A$). Identifying N with this direct summand, it follows that $\|\pi_\tau(a) - \pi_{\text{fin}}(a)\|_{2,\tau} = 0$, which proves the claim.

Being finite, N is the direct sum of a (finite) type I von Neumann algebra and type II_1 von Neuman algebra. We can therefore deal with each summand separately, and combine the two approximations to prove the lemma. To ease the notation, we may as well assume that N itself is type I or type II_1.

First assume N is finite type I, so of the form $N \cong \oplus_i L^\infty(X_i) \otimes M_{n_i}$ for some $n_i \in \mathbb{N}$ and measure spaces X_i. Write $\pi_\tau(a) = \oplus_i \pi_\tau^{(i)}(a)$. If the direct sum is infinite then, by normality of τ, $\pi_\tau(a)$ is the limit in $\|\cdot\|_{2,\tau}$ of the finite sums $\oplus_{i=1}^n \pi_\tau^{(i)}(a)$, and so it suffices to prove the result when the sum $N \cong \oplus_i L^\infty(X_i) \otimes M_{n_i}$ is finite. In this case N is a (non-separable) AF C*-algebra, so given a finite subset $\mathscr{F}$ of the unit ball of N and $\epsilon > 0$ there exists some finite dimensional C*-subalgebra $F \subset N$ such that for each $x \in \mathscr{F}$, there exists a contraction $y_x \in F$ with $\|x - y_x\| < \epsilon$. Fix any conditional expectation $\psi : N \to F$ (an expectation exists by Arveson's Extension Theorem) and note that for $x_1, x_2 \in \mathscr{F}$

$$\begin{aligned}\|\psi(x_1x_2) - \psi(x_1)\psi(x_2)\| &\le \|x_1x_2 - y_{x_1}y_{x_2}\| + \|x_1 - y_{x_1}\| + \|x_2 - y_2\| \\ &\le 4\epsilon. \end{aligned} \tag{2.25}$$

Also, ψ composed with the inclusion map $\phi: F \hookrightarrow N$ is the identity on F, so that $\|\phi(\psi(x)) - x\| \le 2\epsilon$ for $x \in \mathscr{F}$. Thus the required approximations exist in the finite type I case.

Assume now that N is type II_1. The center $Z(N)$ of N is an abelian von Neumann algebra with faithful normal state τ, so of the form $L^\infty(X,\mu)$, where μ is induced by τ. Let $E\colon N \to L^\infty(X,\mu)$ denote the center valued trace. Let $(a_j)_{j=1}^\infty$ be a sequence of positive contractions in A that is dense in the unit ball of A_+ and such that $\|a_j\| < 1$ for all j.

Fix $k \in \mathbb{N}$. Given a k-tuple $i = (i_1, \ldots, i_k) \in \{1, \ldots, k\}^k$, let p_i be the projection in $L^\infty(X,\mu)$, whose characteristic function is the set

$$\{x \in X : \frac{i_j - 1}{k} \leq E(\pi_\tau(a_j))(x) < \frac{i_j}{k},\ j = 1, \ldots, k\}. \tag{2.26}$$

These are pairwise orthogonal and $\sum_i p_i = 1_N$. Some of the p_i may be zero; in what follows we only work with and sum over those indices i for which $p_i \neq 0$. Note that

$$\|E(\pi_\tau(a_j)) - \sum_i \frac{i_j}{k} p_i\|_{L^\infty(X,\mu)} \leq \frac{1}{k},\ j = 1, \ldots, k. \tag{2.27}$$

Now, any normal trace on N is of the form $\tau(f\cdot)$ for some $f \in L^1(X,\mu)_+$ with $\|f\|_{L^1(X,\mu)} = 1$. For such an f,

$$\tau(f\pi_\tau(a_j)) = \tau\big(fE(\pi_\tau(a_j))\big) \approx_{\frac{1}{k}} \sum_i \tfrac{i_j}{k}\tau(fp_i), \quad j = 1, \ldots, k. \tag{2.28}$$

Also, for each index i,

$$|\tau(p_i\pi_\tau(a_j)) - \tau(p_i)\tfrac{i_j}{k}| \leq \frac{1}{k}\tau(p_i), \quad j = 1, \ldots, k. \tag{2.29}$$

Now, for each $i = (i_1, \ldots, i_k)$, the map $\frac{1}{\tau(p_i)}\tau(\pi_\tau(\cdot)p_i)$ is a tracial state on A. Because all traces on A are quasidiagonal, there exist matrix algebras $F_{k,i}$ and u.c.p. maps $\psi_{k,i}\colon A \to F_{k,i}$ such that

$$\left|\mathrm{tr}_{F_{k,i}}\big(\psi_{k,i}(a_j)\big) - \frac{1}{\tau(p_i)}\tau\big(p_i\pi_\tau(a_j)\big)\right| < \frac{1}{k}, \quad j = 1, \ldots, k \tag{2.30}$$

and

$$\|\psi_{k,i}(a_{j_1}a_{j_2}) - \psi_{k,i}(a_{j_1})\psi_{k,i}(a_{j_2})\| < \epsilon, \quad j_1, j_2 = 1, \ldots, k. \tag{2.31}$$

Combining (2.30) and (2.29) gives

$$\left|\mathrm{tr}_{F_{k,i}}\big(\psi_{k,i}(a_j)\big) - \frac{i_j}{k}\right| \leq \frac{2}{k}. \tag{2.32}$$

Define $F_k := \bigoplus_i F_{k,i}$ and $\psi_k := \oplus \psi_{k,i}$ so that (3) holds. Since each $p_i N p_i$ is type II_1, there exists a unital *-homomorphism $\phi_{k,i} : F_{k,i} \to p_i N p_i$ (see e.g. [5, Lemma 2.4.8]). Define $\phi_k : F_k \to N$ by $\phi_k = \oplus_i \phi_{k,i}$. This is a unital *-homomorphism. Further, for each $f \in L^1(X,\mu)_+$ with $\|f\|_{L^1(X,\mu)} = 1$, we have

$$\begin{aligned} \tau\big(f\phi_k(\psi_k(a_j))\big) &= \sum_i \tau(fp_i)\mathrm{tr}_{F_{k,i}}(\psi(a_j)) \\ &\overset{(2.32)}{\approx_{\frac{2}{k}}} \sum_i \tau(fp_i)\tfrac{i_j}{k} \\ &\overset{(2.28)}{\approx_{\frac{1}{k}}} \tau(f\pi_\tau(a_j)), \quad j = 1,\dots,k. \end{aligned} \tag{2.33}$$

Thus the sequence of maps $(\phi_k \circ \psi_k)$ satisfies

$$\lim_{k\to\infty} \sup_{\substack{f\in L^1(X,\mu)_+ \\ \|f\|_{L^1(X,\mu)}=1}} \left|\tau\big(f\phi_k(\psi_k(a_j))\big) - \tau\big(f\pi_\tau(a_j)\big)\right| = 0, \quad j \in \mathbb{N}. \tag{2.34}$$

Write N^ω for the ultraproduct of N with respect to some fixed free ultrafilter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ (defined with respect to τ). We claim that the sequence $(\phi_k \circ \psi_k)$ induces a *-homomorphism, call it $\theta\colon A \to N^\omega$, that agrees on traces with π_τ^ω (the composition of π_τ with the canonical embedding of N into N^ω). Indeed, this follows as in the proof of Lemma 3.21 of [2]: fix j and write $x_{j,k} = \phi_k(\psi_k(a_j)) - \pi_\tau(a_j)$. As $E(x_{j,k} - E(x_{j,k})) = 0$, [13, Theorem 3.2] gives $y_{j,k,l}$ and $z_{j,k,l}$ in N for $l = 1,\dots,10$ such that

$$x_{j,k} - E(x_{j,k}) = \sum_{l=1}^{10} [y_{j,k,l}, z_{j,k,l}] \tag{2.35}$$

with $\|y_{j,k,l}\| \leq 12\|x_{j,k} - E(x_{j,k})\|$ and $\|z_{j,k,l}\| \leq 12$. These estimates ensure that $(y_{j,k,l})_k$ and $(z_{j,k,l})_k$ represent elements $y_{j,l}$ and $z_{j,l}$ in N^ω. Since

$$\|E(x_{j,k})\| = \sup_{\substack{f\in L^1(X,\mu)_+ \\ \|f\|_{L^1(X,\mu)}=1}} \left|\tau\big(f\phi_k(\psi_k(a_j))\big) - \tau\big(f\pi_\tau(a_j)\big)\right|, \tag{2.36}$$

it follows that $(E(x_{j,k}))_k$ represents $0 \in N^\omega$ and so $(x_{j,k})_k$ represents the finite sum of commutators $\sum_{l=1}^{10}[y_{j,l}, z_{j,l}]$ in N^ω and hence is zero in all traces on N^ω.

By the remark preceding the lemma, θ and π_τ^ω are σ-strong* approximately unitarily equivalent. Because A is separable and we work in an ultrapower, a standard reindexing argument (using Kirchberg's ϵ-test from [20, Appendix A]) shows that θ and π_τ^ω are actually unitarily equivalent. That is, there exists a sequence

(u_k) of unitaries in N such that

$$\lim_{k\to\omega} \|u_k(\phi_k \circ \psi_k)(a)u_k^* - \pi_\tau(a)\|_{2,\tau} = 0, \quad a \in A. \tag{2.37}$$

Let $\tilde{\phi}_k = \mathrm{Ad}\, u_k \circ \phi_k$. Passing to a subsequence, if necessary, we obtain

$$\lim_{k\to\infty} \|(\tilde{\phi}_k \circ \psi_k)(a) - \pi_\tau(a)\|_{2,\tau} = 0, \quad a \in A, \tag{2.38}$$

as was to be proved. □

Proof of Proposition 2.3 For unital C^*-algebras, one just takes direct sums of the maps provided by Lemmas 2.4 and 2.5. The non-unital case follows from the unital case as follows.

Assume A is non-unital and $T(A) = T_{\mathrm{qd}}(A)$. Then by [6, Proposition 3.5.10] we have $T(\tilde{A}) = T_{\mathrm{qd}}(\tilde{A})$, too, where $\tilde{A}$ is the unitization of A. Hence we can find nets of finite-dimensional C^*-algebras (F_i) and c.p.c. maps

$$\tilde{A} \xrightarrow{\psi_i} F_i \xrightarrow{\phi_i} (\tilde{A})^{**} \tag{2.39}$$

such that

1. $(\phi_i \circ \psi_i)(a) \to \iota_{\tilde{A}}(a)$, in the σ-weak topology for all $a \in \tilde{A}$
2. every ϕ_i is a convex combination of finitely many contractive order zero maps; and
3. $\|\psi_i(ab) - \psi_i(a)\psi_i(b)\| \to 0$ for all $a, b \in \tilde{A}$.

The short exact sequence $0 \to A \to \tilde{A} \to \mathbb{C} \to 0$ induces a canonical isomorphism $(\tilde{A})^{**} \cong A^{**} \oplus \mathbb{C}$. The desired maps are now gotten by restricting each ψ_i to A and using the σ-weakly continuous projection $(\tilde{A})^{**} \to A^{**}$ to push the ϕ_i's back into A^{**}. □

3 The Main Theorem

Theorem 3.1 *Let A be a nuclear C^*-algebra. Then there exist nets of finite-dimensional C^*-algebras (F_i) and c.p.c. maps*

$$A \xrightarrow{\psi_i} F_i \xrightarrow{\phi_i} A \tag{3.1}$$

such that

1. $\|(\phi_i \circ \psi_i)(a) - a\| \to 0$ *for all* $a \in A$;
2. *every* ϕ_i *is a convex combination of finitely many contractive order zero maps;*
3. $\|\psi_i(a)\psi_i(b)\| \to 0$ *for all* $a, b \in A_+$ *that satisfy* $ab = 0$.

To prove Theorem 3.1 we will apply Theorem 2.2 to the cone $CA = C_0(0,1] \otimes A$ of A. We will need to know that all traces on CA are quasidiagonal for nuclear A. While this follows from [24, Corollary 6.1],[3] it is really the case that the required statement is a recasting of the "order zero quasidiagonality result" of [22, Proposition 3.2] used as the starting point in [24]. More generally, Gabe's "order zero quasidiagonality" of amenable traces [14, Proposition 3.5] can also be expressed in this language, as set out below.

Proposition 3.2 (Gabe, cf. [14, Proposition 3.5]) *Let A be a C^*-algebra. Then every amenable trace on CA is quasidiagonal. In particular if A is nuclear, then all traces on CA are quasidiagonal.*

Proof It is well known that traces of the form $\delta_t \otimes \tau_A$, where δ_t is evaluation at some $t \in (0,1]$ and τ_A is a trace on A, generate the Choquet simplex of traces on the cone CA.[4] Since the amenable traces on CA form a face ([19, Lemma 3.4], see also [5, Proposition 6.3.7]) and the set of quasidiagonal traces is a weak*-closed, convex subset of $T(A)$ [6, Proposition 3.5.1], it suffices to show that any amenable trace on CA of the form $\delta_t \otimes \tau_A$ for some $t \in (0,1]$ and some trace τ_A on A is quasidiagonal.

Note too that if $\delta_t \otimes \tau_A$ is an amenable trace on CA, then τ_A is amenable on A. This follows from [6, Theorem 3.1.6] by checking that the tensor product functional μ_{τ_A} on the algebraic tensor product $A \odot A^{\mathrm{op}}$ given by $\mu_{\tau_A}(a \otimes b^{\mathrm{op}}) = \tau_A(ab)$ is continuous with respect to the minimal tensor product. Let $g \in C_0(0,1]$ be a positive contraction with $g(t) = 1$. Then μ_{τ_A} factorizes as

$$A \odot A^{\mathrm{op}} \xrightarrow{a\otimes b^{\mathrm{op}} \mapsto (g\otimes a)\otimes(g\otimes b)^{\mathrm{op}}} CA \odot (CA)^{\mathrm{op}} \xrightarrow{\mu_{\delta_t \otimes \tau_A}} \mathbb{C}; \tag{3.2}$$

the first of these maps is the tensor product of two c.p.c. maps, so contractive with respect to the minimal tensor product, while contractivity of $\mu_{\delta_t \otimes \tau_A}$ follows from the assumption that $\delta_t \otimes \tau_A$ is amenable.

At this point, if A is not unital, then we can unitize A, and τ_A (since the unitization of an amenable trace remains amenable). As a final reduction, by considering the map $CA \to C_0((0,t], A)$ given by restriction, and then identifying $C_0((0,t], A)$ with CA (by rescaling), we may as well assume that $t = 1$. Then [14, Proposition 3.5] gives a c.p.c. order zero map $\Psi : A \to \mathcal{Q}_\omega$ (where $\mathcal{Q}$ denotes the universal UHF algebra and $\mathcal{Q}_\omega$ its ultrapower) such that

$$\tau_{\mathcal{Q}_\omega}\big(\Psi(a)\Psi(1_A)^{n-1}\big) = \tau_A(a), \quad a \in A,\ n \in \mathbb{N}. \tag{3.3}$$

By the correspondence between order zero maps from A and *-homomorphisms from CA (see [28, Corollary 4.1]) we obtain a *-homomorphism $\psi\colon CA \to \mathcal{Q}_\omega$ such

[3]The cone CA is quasidiagonal by [25] and satisfies the UCT, since it is contractible.

[4]That is, any trace on CA lies in the weak*-closed convex hull of the specified traces.

that $\psi(\mathrm{id}_{(0,1]} \otimes a) = \Psi(a)$ for every $a \in A$. Then for every $a \in A$ and $n \in \mathbb{N}$,

$$\begin{aligned}\tau_{\mathcal{Q}_\omega}\big(\psi(\mathrm{id}^n_{(0,1]} \otimes a)\big) &= \tau_{\mathcal{Q}_\omega}\big(\Psi(a)\Psi(1_A)^{n-1}\big) \\ &= \tau_A(a) = (\delta_1 \otimes \tau_A)(\mathrm{id}^n_{(0,1]} \otimes a). \qquad (3.4)\end{aligned}$$

Thus ψ witnesses the quasidiagonality of the trace $\delta_1 \otimes \tau_A$. □

Proof of Theorem 3.1 Let $\mathcal{F} \subset A$ be finite and $\epsilon > 0$. Then there is a separable nuclear subalgebra B of A containing $\mathcal{F}$. Write $\iota : B \to A$ for the canonical inclusion map.

Let $\theta : B \to CB$ be the c.p.c. order zero map $b \mapsto \mathrm{id}_{(0,1]} \otimes b$. Notice that CB satisfies the hypotheses of Theorem 2.2: it is certainly separable and nuclear, it is quasidiagonal by a theorem of Voiculescu [25], and all of its traces are quasidiagonal by Proposition 3.2 (as CB is nuclear, all traces are amenable). Then there are a finite dimensional algebra F and c.p.c maps $\psi : CB \to F$ and $\phi : F \to CB$ such that

1. $\|(\phi \circ \psi)(\theta(x)) - \theta(x)\| < \epsilon$;
2. ϕ is a convex combination of finitely many contractive order zero maps; and
3. $\|\psi\big(\theta(x)\theta(y)\big) - \psi(\theta(x))\psi(\theta(y))\| < \epsilon$;

for all $x, y \in \mathcal{F}$. Let $\eta : CB \to B$ be given by the point evaluation at 1 so that $\eta \circ \theta = \mathrm{id}_B$.

Define a c.p.c. map $\overline{\psi} : A \to F$ by extending $\psi \circ \theta$ to A (using Arveson's extension theorem) and set $\overline{\phi} = \iota \circ \eta \circ \phi : F \to A$. Then $\overline{\phi}$ is a convex combination of contractive order zero maps (because $\iota \circ \eta$ is a *-homomorphism), $\|(\overline{\phi} \circ \overline{\psi})(x) - x\| < \epsilon$ for every $x \in \mathcal{F}$, and $\|\overline{\psi}(x)\overline{\psi}(y)\| < \epsilon$ if $x, y \in \mathcal{F}$ are orthogonal positive elements. □

Remark 3.3 As with the approximations in [16], attempting to merge the approximations of Theorem 3.1 with the nuclear dimension by additionally asking for a uniform bound on the number of summands in the decompositions of Φ_i as a convex combination of order zero maps is very restrictive. By the main result of [8], such approximations only exist for AF C^*-algebras.

Acknowledgements S.W. would like to thank Ilan Hirshberg for many helpful conversations regarding approximations of nuclear C*-algebras and the organisers of the Abel symposium for a fantastic conference. The authors would also like to thank the referee for a number of helpful suggestions and comments.

N.B. was partially supported by NSF grant DMS-1201385; J.C. by NSF Postdoctoral Fellowship DMS-1303884; S.W. by an Alexander von Humboldt foundation fellowship and by the DFG (SFB 878).

References

1. Scott Atkinson, *Convex sets associated to* C**-algebras*, J. Funct. Anal., to appear, arXiv:1509.00822, 2015.
2. Joan Bosa, Nathanial P. Brown, Yasuhiko Sato, Aaron Tikuisis, Stuart White, and Wilhelm Winter, *Covering dimension of* C**-algebras and 2-coloured classification*, Mem. Amer. Math. Soc., to appear, arXiv:1506.03974, 2015.
3. Bruce Blackadar and Eberhard Kirchberg, *Generalized inductive limits of finite-dimensional* C**-algebras*, Math. Ann. **307** (1997), no. 3, 343–380.
4. Bruce Blackadar, *Operator algebras*, Encyclopaedia of Mathematical Sciences, vol. 122, Springer-Verlag, Berlin, 2006, Theory of C^*-algebras and von Neumann algebras, Operator Algebras and Non-commutative Geometry, III.
5. Nathanial P. Brown and Narutaka Ozawa, C*-algebras and finite-dimensional approximations, Graduate Studies in Mathematics, vol. 88, American Mathematical Society, Providence, RI, 2008.
6. Nathanial P. Brown, *Invariant means and finite representation theory of* C**-algebras*, Mem. Amer. Math. Soc. **184** (2006), no. 865, viii+105.
7. Nathanial P. Brown and Wilhelm Winter, *Quasitraces are traces: a short proof of the finite-nuclear-dimension case*, C. R. Math. Acad. Sci. Soc. R. Can. **33** (2011), no. 2, 44–49.
8. Jorge Castillejos, *Decomposable approximations and approximately finite dimensional C^*-algebras*, Math. Proc. Cam. Phil. Soc., to appear, arXiv:1409.7304.
9. Alain Connes, *Classification of injective factors. Cases II_1, II_∞, III_λ, $\lambda \neq 1$*, Ann. of Math. (2), **104**, (1976), no. 1, 73–115.
10. Man Duen Choi and Edward G. Effros, *Nuclear* C**-algebras and the approximation property*, Amer. J. Math. **100** (1978), no. 1, 61–79.
11. Huiru Ding and Don Hadwin, *Approximate equivalence in von Neumann algebras*, Sci. China Ser. A **48** (2005), no. 2, 239–247.
12. George A. Elliott, *On approximately finite-dimensional von Neumann algebras. II*, Canad. Math. Bull. **21** (1978), no. 4, 415–418.
13. Thierry Fack and Pierre de la Harpe, *Sommes de commutateurs dans les algèbres de von Neumann finies continues*, Ann. Inst. Fourier (Grenoble) **30** (1980), no. 3, 49–73.
14. James Gabe, *Quasidiagonal traces on exact* C**-algebras*, arXiv:1511.02760, 2015.
15. Uffe Haagerup, *A new proof of the equivalence of injectivity and hyperfiniteness for factors on a separable Hilbert space*, J. Funct. Anal. **62** (1985), no. 2, 160–201.
16. Ilan Hirshberg, Eberhard Kirchberg, and Stuart White, *Decomposable approximations of nuclear* C**-algebras*, Adv. Math. **230** (2012), no. 3, 1029–1039.
17. Kenley Jung, *Amenability, tubularity, and embeddings into $\mathscr{R}^\omega$*, Math. Ann. **338** (2007), no. 1, 241–248.
18. Eberhard Kirchberg, C*-nuclearity implies CPAP, Math. Nachr. **76** (1977), 203–212.
19. ______, *Discrete groups with Kazhdan's property* T *and factorization property are residually finite*, Math. Ann. **299** (1994), no. 3, 551–563.
20. ______, *Central sequences in* C**-algebras and strongly purely infinite algebras*, Operator Algebras: The Abel Symposium 2004, Abel Symp., vol. 1, Springer, Berlin, 2006, pp. 175–231
21. Leonel Robert, *Nuclear dimension and n-comparison*, Münster J. Math. **4** (2011), 65–71.
22. Yasuhiko Sato, Stuart White, and Wilhelm Winter, *Nuclear dimension and $\mathscr{Z}$-stability*, Invent. Math. **202** (2015), no. 2, 893–921.
23. Masamichi Takesaki, *Theory of operator algebras. III*, Encyclopaedia of Mathematical Sciences, vol. 127, Springer-Verlag, Berlin, 2003, Operator Algebras and Non-commutative Geometry, 8.
24. Aaron Tikuisis, Stuart White, and Wilhelm Winter, *Quasidiagonality of nuclear* C**-algebras*, arXiv:1509.08318, 2015.

25. Dan Voiculescu, *A note on quasi-diagonal* C*-*algebras and homotopy*, Duke Math. J. **62** (1991), no. 2, 267–271.
26. Wilhelm Winter, *Decomposition rank and $\mathscr{Z}$-stability*, Invent. Math. **179** (2010), no. 2, 229–301.
27. ———, *Nuclear dimension and $\mathscr{Z}$-stability of pure* C*-*algebras*, Invent. Math. **187** (2012), no. 2, 259–342.
28. Wilhelm Winter and Joachim Zacharias, *Completely positive maps of order zero*, Münster J. Math. **2** (2009), 311–324.
29. ———, *The nuclear dimension of* C*-*algebras*, Adv. Math. **224** (2010), no. 2, 461–498.

Exotic Crossed Products

Alcides Buss, Siegfried Echterhoff, and Rufus Willett

Abstract An exotic crossed product is a way of associating a C^*-algebra to each C^*-dynamical system that generalizes the well-known universal and reduced crossed products. Exotic crossed products provide natural generalizations of, and tools to study, exotic group C^*-algebras as recently considered by Brown-Guentner and others. They also form an essential part of a recent program to reformulate the Baum-Connes conjecture with coefficients so as to mollify the counterexamples caused by failures of exactness.

In this paper, we survey some constructions of exotic group algebras and exotic crossed products. Summarising our earlier work, we single out a large class of crossed products—the *correspondence functors*—that have many properties known for the maximal and reduced crossed products: for example, they extend to categories of equivariant correspondences, and have a compatible descent morphism in KK-theory. Combined with known results on K-amenability and the Baum-Connes conjecture, this allows us to compute the K-theory of many exotic group algebras. It also gives new information about the reformulation of the Baum-Connes Conjecture mentioned above. Finally, we present some new results relating exotic crossed products for a group and its closed subgroups, and discuss connections with the reformulated Baum-Connes conjecture.

A. Buss
Departamento de Matemática, Universidade Federal de Santa Catarina, 88.040-900 Florianópolis-SC, Brazil
e-mail: alcides@mtm.ufsc.br

S. Echterhoff (✉)
Mathematisches Institut, Westfälische Wilhelms-Universität Münster, Einsteinstr. 62, 48149 Münster, Germany
e-mail: echters@uni-muenster.de

R. Willett
University of Hawai‘i at Mānoa, Keller 401A, 2565 McCarthy Mall, Honolulu, HI 96822, USA
e-mail: rufus@math.hawaii.edu

T.M. Carlsen et al. (eds.), *Operator Algebras and Applications*, Abel Symposia 12,
DOI 10.1007/978-3-319-39286-8_3

1 Introduction

Given a C*-dynamical system (A, G, α), there are two classical ways to assign a C*-algebra to it which reflect important properties of the system: the universal (or maximal) crossed product $A \rtimes_{\alpha,\mathrm{u}} G$, which is universal for covariant representations, and the reduced crossed product $A \rtimes_{\alpha,\mathrm{r}} G$ defined as the image of $A \rtimes_{\alpha,\mathrm{u}} G$ under the regular representation. If $A = \mathbb{C}$ we recover the maximal and reduced group C*-algebras $C^*_{\mathrm{u}}(G)$ and $C^*_{\mathrm{r}}(G)$ of the group G.

Recently there has been a growing interest in studying more general "exotic" group algebras $C^*_\mu(G)$ and crossed products $A \rtimes_{\alpha,\mu} G$ which are defined as completions of the convolution algebra $C_c(G)$ (resp. $C_c(G, A)$) with respect to C*-norms which lie between the full and reduced norms $\|\cdot\|_{\mathrm{u}}$ and $\|\cdot\|_{\mathrm{r}}$. For exotic group algebras, this started with the work [4] of Brown and Guentner: given a discrete group G and an (algebraic) ideal E in $l^\infty(G)$, Brown and Guentner assign an exotic group algebra $C^*_{E,BG}(G)$ by looking at representations of G that have 'enough' matrix coefficients in E. Later work of Okayasu [32] shows that this theory is extremely rich: If G is a discrete group which contains the free group F_2 on two generators as a subgroup, then there are uncountably many different exotic group algebras $C^*_E(G)$ of this type. Wiersma [36] subsequently extended the framework of Brown and Guentner to general locally compact groups, and showed an analogous result to Okayasu's for the connected group $SL(2, \mathbb{R})$. Motivated by the work of Brown and Guentner, the paper [25] of Kaliszewski, Landstad and Quigg introduced a second construction of exotic group algebras based on the dual pairing of $C^*_{\mathrm{u}}(G)$ with the Fourier-Stieltjes algebra $B(G)$, and studied the connection to coactions. We shall give a detailed introduction to exotic group algebras in Sect. 2 below and we use this opportunity to clarify the differences between the approaches of Brown-Guentner and of Kaliszewski-Landstad-Quigg.

Work on exotic crossed products starts also with Brown-Guentner [4] and Kaliszewski-Landstad-Quigg [25] who both indicated ways to associate exotic crossed-products $(A, \alpha) \mapsto A \rtimes_{\alpha,\mu_E} G$ to appropriate spaces of matrix coefficients E. The constructions of exotic group algebras of Brown-Guentner and of Kaliszewski-Landstad-Quigg are closely related, but their constructions of exotic crossed products are fundamentally quite different in nature; nonetheless, it was observed later [3, 6] that both constructions are functorial in the sense that every G-equivariant *-homomorphism $\Phi : A \to B$ between two G-algebras (A, α) and (B, β) descends to a *-homomorphism $\Phi \rtimes_{\mu_E} G : A \rtimes_{\alpha,\mu_E} G \to B \rtimes_{\beta,\mu_E} G$ in a canonical way.

Such exotic crossed product *functors* were formally introduced by Baum, Guentner, and Willett in [3] as part of a program to "fix" the Baum-Connes conjecture for computing the K-theory of crossed products. The original conjecture (in the general version with coefficients) claimed that for each system (A, G, α) a certain *assembly map*

$$\mathrm{as}^{\mathrm{r}}_{(G,A)} : K^{\mathrm{top}}_*(G; A) \to K_*(A \rtimes_{\alpha,\mathrm{r}} G)$$

should always be an isomorphism, where $K_*^{\mathrm{top}}(G;A)$ is the *topological K-theory of G with coefficient A* (we refer to [2] for the construction of this group and for a discussion of the far reaching consequences for groups G that satisfy the conjecture). Unfortunately, the conjecture is not true for all groups: the known counterexamples are all related to the failure of G being exact in the sense of Kirchberg and Wassermann. In order to fix the conjecture, the main idea of Baum, Guentner, and Willett was to replace the reduced crossed-product functor by the *smallest exact Morita compatible* crossed-product functor $(A,\alpha) \mapsto A \rtimes_{\alpha,\mathscr{E}} G$. Here Morita compatibility basically means that the construction preserves *stabilisation* which is needed to construct an E-theory descent used for the construction of a direct assembly map

$$\mathrm{as}^{\mathscr{E}}_{(G,A)} : K_*^{\mathrm{top}}(G;A) \to K_*(A \rtimes_{\alpha,\mathscr{E}} G);$$

exactness and Morita compatibility are both predicted on the level of K-theory by the conjecture, so are natural assumptions. The new conjecture asserts that this map is always an isomorphism. In fact, Baum, Guentner, and Willett show in [3] that several of the known counterexamples for the original conjecture satisfy the new conjecture and, so far, there are no known counterexamples for the reformulated conjecture.

Motivated by the above described developments, the authors of this paper started in [9] a more systematic study of various functorial properties of exotic crossed-product functors $(A,\alpha) \mapsto A \rtimes_{\alpha,\mu} G$. Recall first that a correspondence between two C*-algebras A and B consists of a Hilbert B-module $\mathscr{E}_B$ equipped with a left action $\Phi : A \to \mathscr{L}(\mathscr{E}_B)$ of A on $\mathscr{E}_B$. In case of the universal or reduced crossed-products $\rtimes_{\mathrm{u}}$ or $\rtimes_{\mathrm{r}}$, respectively, it has been known for a long time that they are

1. functorial for *generalised homomorphisms* in the sense that any G-equivariant *-homomorphism $\Phi : A \to \mathscr{M}(B)$ canonically induces *-homomorphisms
$$\Phi \rtimes_\mu G : A \rtimes_{\alpha,\mu} G \to \mathscr{M}(B \rtimes_{\beta,\mu} G), \quad \text{if } \mu = \mathrm{u} \text{ or } \mu = \mathrm{r}.$$
2. functorial for correspondences: This means that any G-equivariant $A - B$ correspondence $\mathscr{E}_B$ admits a *descent* $\mathscr{E}_B \rtimes_\mu G$ which is an $A \rtimes_{\alpha,\mu} G - B \rtimes_{\beta,\mu} G$ correspondence and this construction preserves composition of correspondences.
3. functorial for Kasparov's KK-theory: This simply means that there is a descent homomorphism
$$j_G^\mu : KK^G(A,B) \to KK(A \rtimes_{\alpha,\mu} G, B \rtimes_{\beta,\mu} G), \quad \text{if } \mu = \mathrm{u} \text{ or } \mu = \mathrm{r},$$
which is compatible with taking Kasparov products.

In Sects. 3–5 we give a survey of the general theory of exotic crossed products and, in particular, the results obtained in [9] in which we give characterisations of crossed products which enjoy strong functorial properties as described above.

As a sample, it turned out that functoriality for generalised homomorphisms is equivalent to the *ideal property* which asserts that for each G-invariant ideal $I \subseteq A$ the inclusion $\iota : I \hookrightarrow A$ descends to a faithful morphism $\iota \rtimes_\mu G : I \rtimes_\mu G \hookrightarrow A \rtimes_\mu G$. Similarly, functoriality for correspondences is equivalent to the *projection property* which asserts that for every G-invariant projection $p \in \mathcal{M}(A)$ the inclusion $\iota : pAp \hookrightarrow A$ descends to an inclusion $\iota \rtimes_\mu G : pAp \rtimes_\mu G \hookrightarrow A \rtimes_\mu G$. Another useful characterisation of correspondence functors is functoriality with respect to G-equivariant completely positive maps, and as an application of this we show here as a new result that correspondence crossed products behave well with respect to taking tensor products with nuclear C*-algebras: If $\rtimes_\mu$ is a correspondence functor, then

$$(A \otimes B) \rtimes_{\alpha\otimes \mathrm{id}_B,\mu} G \cong (A \rtimes_{\alpha,\mu} G) \otimes B$$

for any nuclear C*-algebra B. As an application, we can show that under suitable assumptions it turns out that any crossed-product functor which is faithful in the sense that it does not send nonzero objects to $\{0\}$ must dominate the reduced crossed product functor. This motivates the requirement that exotic crossed products lie between the maximal and the reduced ones.

It is straightforward to see that the functors constructed by Brown-Guentner and Kaliszewski-Landstad-Quigg (which we shall call *BG-functors* or *KLQ-functors*, respectively) satisfy the ideal property. Moreover, all KLQ-functors are correspondence functors, but a BG-functor is a correspondence functor if and only if it coincides with the universal crossed-product functor.

It was shown in [9] that every correspondence functor admits a descent in KK-theory. Adapting ideas of Cuntz and Julg-Valette from [11] and [23] this fact has then been used to show that for any K-amenable group in the sense of Cuntz and for any correspondence functor $\rtimes_\mu$, the canonical morphisms

$$A \rtimes_{\alpha,u} G \twoheadrightarrow A \rtimes_{\alpha,\mu} G \twoheadrightarrow A \rtimes_{\alpha,\mathrm{r}} G$$

are KK-equivalences. In particular, this implies that for $G = F_n$, the free group in n generators, $n \in \mathbb{N} \cup \{\infty\}$, all of the uncountably many different exotic group algebras $C^*_E(F_n)$ corresponding to nontrivial G-invariant ideals $E \subseteq B(G)$ are KK-equivalent. The result would be false without the assumption that E is an ideal, and indeed it is crucial for us that an exotic group algebra is of the form $\mathbb{C} \rtimes_\mu G$ for a correspondence functor $\rtimes_\mu$ if and only if it is of the form $C^*_E(G)$ for a G-invariant weak*-closed ideal E in $B(G)$.

We also report on the result that the minimal exact Morita compatible functor $\rtimes_{\mathscr{E}}$ coincides, at least on separable systems, with the minimal exact correspondence functor $\rtimes_{\mathscr{E}_{\mathfrak{Corr}}}$, whose existence is shown in [9]. Hence our results allow one to use the full force of Kasparov's equivariant KK-theory for the study of the reformulated Baum-Connes conjecture.

In the remaining sections (Sects. 6–8) we report on some new results about the relation of crossed-product functors of a group G with crossed-product functors

of closed subgroups H of G. Based on Green's imprimitivity theorem we give a procedure of restricting crossed-product functors from G to H. Conversely, given a crossed-product functor $\rtimes_\nu$ for H, we describe two different ways to assign a crossed-product functor for G to it: one is the induced functor $\rtimes_{\mathrm{Ind}\,\nu}$ and one is the extended functor $\rtimes_{\mathrm{ext}\nu}$. All these constructions preserve the ideal property (hence functoriality for generalised morphisms) and send correspondence functors to correspondence functors. Restriction and extension preserve exactness in general, but this can fail for induction. After introducing the restriction, induction and extension procedures in Sect. 6 we show in Sect. 7 that the restriction of the minimal exact correspondence functor $\rtimes_{\mathscr{E}^G_{\mathfrak{Corr}}}$ of G to a closed *normal* subgroup N of G always coincides with the minimal exact correspondence functor $\rtimes_{\mathscr{E}^N_{\mathfrak{Corr}}}$ of N. Using results of Chabert and Echterhoff from [10] this implies that the validity of the reformulated Baum-Connes conjecture for G always passes to closed normal subgroups $N \subseteq G$. Another result in this direction (obtained in Sect. 6), shows that for a cocompact closed subgroup $H \subseteq G$ the induction of $\rtimes_{\mathscr{E}^H_{\mathfrak{Corr}}}$ to G coincides with $\rtimes_{\mathscr{E}^G_{\mathfrak{Corr}}}$.

Finally, in Sect. 8, we close this paper with some questions and remarks related to induction, restriction and extension in connection with more general permanence properties of the reformulated Baum-Connes conjecture. Unfortunately, so far it seems that we have more questions than solutions!

2 Exotic Group Algebras

Let G be a locally compact group equipped with a fixed left invariant Haar measure. We are interested in C^*-algebras connected to strongly continuous unitary representations $u : G \to \mathscr{U}(\mathscr{H})$ of G, which we call simply *representations* of G.

Let $C_c(G)$ denote the space of continuous, compactly supported, complex-valued functions on G, which is a *-algebra when equipped with the product and involution

$$f * g(t) := \int_G f(s)g(s^{-1}t)\,ds \quad \text{and} \quad f^*(t) := \Delta(t)^{-1}\overline{f(t^{-1})},$$

where $\Delta : G \to (0,\infty)$ is the modular function. Any representation $u : G \to \mathscr{U}(\mathscr{H})$ integrates to a *-representation $\tilde{u} : C_c(G) \to \mathscr{L}(\mathscr{H})$ defined by

$$\tilde{u}(f) := \int_G f(s)u_s\,ds.$$

In what follows we shall usually omit the tilde in our notations, thus using the same notation for the unitary representation u and its integrated form.

The following two completions of $C_c(G)$ are intimately tied to the representation theory of G, and for this and other reasons have been widely studied.

Definition 2.1 The *universal group algebra* $C^*_{\mathrm{u}}(G)$ is the completion of $C_c(G)$ with respect to the norm

$$\|f\|_{\mathrm{u}} := \sup\{\|u(f)\| : u \text{ a representation of } G\}.$$

Let $\lambda : G \to \mathcal{U}(L^2(G))$ be the left regular representation: $\lambda_s(\xi)(t) := \xi(s^{-1}t)$. The *reduced group algebra* $C^*_{\mathrm{r}}(G)$ is the completion of $C_c(G)$ with respect to the norm

$$\|f\|_{\mathrm{r}} := \|\lambda(f)\|.$$

Remark 2.2 As every C^*-algebra A admits a faithful nondegenerate representation on some Hilbert space, every strictly continuous homomorphism $u\colon G \to \mathcal{U}\mathcal{M}(A)$ integrates uniquely to a nondegenerate *-homomorphism $C^*_{\mathrm{u}}(G) \to \mathcal{M}(A)$. Conversely, every such homomorphism is the integrated form of a representation of G and this universal property characterises $C^*_{\mathrm{u}}(G)$ up to isomorphism.

Definition 2.3 A *group algebra* $C^*_\mu(G)$ is any C^*-algebra completion of $C_c(G)$ for a C^*-norm $\|\cdot\|_\mu$ such that for all $f \in C_c(G)$

$$\|f\|_{\mathrm{u}} \geq \|f\|_\mu \geq \|f\|_{\mathrm{r}}.$$

A group algebra $C^*_\mu(G)$ is *exotic* if $\|\cdot\|_\mu$ is not equal to either the maximal or reduced norms.

It is perhaps not immediately clear that interesting examples exist! There have been two recent approaches to this in the literature that we discuss below, the first due to Brown and Guentner [4], and the second due to Kaliszewski, Landstad, and Quigg [24, Section 3]. We first discuss the construction of Brown-Guentner, and then describe theorems of Okayasu and Wiersma that show that the theory of exotic group algebras is very rich.

Definition 2.4 Let D be a set of complex-valued functions on G. A representation $u : G \to \mathcal{U}(\mathcal{H})$ of G is called a *D-representation* if there exists a dense subspace $\mathcal{H}_0$ of $\mathcal{H}$ such that for all $\xi, \eta \in \mathcal{H}_0$, the *matrix coefficient*

$$G \to \mathbb{C}, \quad g \mapsto \langle \xi, u_g \eta \rangle$$

is in D.[1] Define a C^*-seminorm on $C_c(G)$ by

$$\|f\|_{D,BG} := \sup\{\|u(f)\| \mid u \text{ a } D\text{-representation}\},$$

and then define $C^*_{D,BG}(G)$ to be the Hausdorff completion of $C_c(G)$ for the seminorm $\|\cdot\|_{D,BG}$.

[1] We always assume inner products on Hilbert spaces to be linear in the *second* variable.

Remark 2.5

1. In the original version of their definition [4, Definition 2.2], Brown and Guentner work only with discrete groups, but their definition extends in an obvious way to all locally compact groups as already pointed out by Wiersma [36, Section 3]. They moreover assume that D is an algebraic ideal in the space $l^\infty(G)$; this is important for their applications. It is clear, however, that the definition makes sense without these additional assumptions; we left them out as the extra generality seems harmless, and is occasionally useful.
2. Let $B(G)$ denote the *Fourier-Stieltjes algebra* of G: the space of all matrix coefficients of G, which turns out to be a *-algebra of bounded functions on G for the natural pointwise operations; we will have more to say on this below. The C^*-algebra $C^*_{D,BG}(G)$ clearly only depends on the intersection $D \cap B(G)$. It is often convenient to pass to this intersection as it is the algebraic and topological properties of $D \cap B(G)$ as a subset of the algebra $B(G)$ that are really relevant for the structure of $C^*_{D,BG}(G)$. For example, when Brown and Guentner use that D is an ideal in $l^\infty(G)$ for certain of their applications, what is really relevant is that $D \cap B(G)$ is an ideal in $B(G)$; the latter is a strictly weaker property. On the other hand, the passage from $l^\infty(G)$ to $B(G)$ also loses some concreteness, as it is generally difficult to tell when a given function is in $B(G)$.
3. The completion $C^*_{D,BG}(G)$ is a group C^*-algebra in our sense if and only if $\|\cdot\|_{D,BG}$ dominates the reduced norm. For this, it is sufficient (for example) that D contains all compactly supported functions in $B(G)$.

We now turn to a natural class of examples based on decay of matrix coefficients. For $p \in [1,\infty)$, write $C^*_p(G)$ as shorthand for $C^*_{L^p(G),BG}(G)$. Note that if $p < q$ then $L^p(G) \cap B(G) \subseteq L^q(G) \cap B(G)$ and thus the identity map on $C_c(G)$ extends to a surjective *-homomorphism

$$C^*_q(G) \twoheadrightarrow C^*_p(G). \tag{2.1}$$

The following theorem is due to Okayasu for F_2 [32] and Wiersma for $SL(2,\mathbb{R})$ [36]: it shows in particular that the construction above gives rise to many interesting exotic group algebras.

Theorem 2.6 *Say G is either the free group F_2 on two generators, or $SL(2,\mathbb{R})$. Let $p > q \geq 2$. Then the canonical quotient map in line* (2.1) *above is not injective.*

Remark 2.7

1. All the C^*-algebras $C^*_p(SL(2,\mathbb{R}))$ for $p \in (2,\infty)$ are in fact *abstractly* isomorphic, as one can see by combining Wiersma's analysis [36] with Miličić's description of the structure of $C^*_{\mathrm{u}}(SL(2,\mathbb{R}))$ [31]. We do not know if $C^*_p(F_2)$ is abstractly isomorphic to $C^*_q(F_2)$ for any distinct p,q in $(2,\infty)$.
2. The key ingredients in the proof of Theorem 2.6 are quite deep facts from harmonic analysis: for F_2 the result relies on Haagerup's study of $C^*_{\mathrm{r}}(F_2)$ in [18], while for $SL(2,\mathbb{R})$ the proof relies on aspects of the Kunze-Stein phenomenon [30].

3. The result of Okayasu extends to all (non-amenable) discrete groups that contain F_2 as a subgroup, and conceivably to all non-amenable discrete groups. However, Wiersma's result certainly does not extend to all non-amenable connected groups. Indeed for any $n > 2$, it follows from results of Scaramuzzi [34, Theorem III.3.3] (see also the discussion in [19, Section V.3.3]) that if $G = SL(n, \mathbb{R})$, then $C^*_p(G) = C^*_q(G)$ for all $p, q \in (n, \infty)$.

Having established that many interesting exotic group algebras exist, let us turn to the construction of Kaliszewski-Landstad-Quigg. This gives another perspective that will be especially convenient when we come to discuss exotic crossed products. In order to do this, we need a little more background on the Fourier-Stieltjes algebra $B(G)$. Recall that $B(G)$ consists of all matrix coefficients of G, that is, functions $\phi : G \to \mathbb{C}$ of the form $\phi(g) = \langle \xi, u_g \eta \rangle$ for some representation $u : G \to \mathscr{U}(\mathscr{H})$ and vectors $\xi, \eta \in \mathscr{H}$; note that elements of $B(G)$ are necessarily bounded and continuous. Straightforward algebraic checks based on the fact that one can take direct sums, contragredients, and tensor products of representations show that $B(G)$ is a *-algebra under the usual pointwise operations. Note also that $B(G)$ is invariant under the G-actions on functions induced by the left and right translation actions of G on itself. For brevity, we say that a collection E of functions on G is *translation invariant* if it is preserved by the actions on functions induced by the left and right translation actions of G on itself.

There is a pairing between $C_c(G)$ and $B(G)$ defined by

$$\langle f, \phi \rangle := \int_G f(s)\phi(s)\, ds, \quad f \in C_c(G), \phi \in B(G).$$

This clearly extends to a bilinear pairing between $C^*_{\mathrm{u}}(G)$ and $B(G)$. Moreover, it follows from the identification of unitary representations of G and *-representations of $C^*_{\mathrm{u}}(G)$, together with the GNS construction, that this pairing identifies $B(G)$ with the dual space $C^*_{\mathrm{u}}(G)'$ of $C^*_{\mathrm{u}}(G)$. We equip $B(G)$ with the norm, and also weak* topology, coming from this identification. See [15] for more information on $B(G)$.

Let now $\mathscr{U}(\mathscr{M}(C^*_{\mathrm{u}}(G)))$ be the unitary group of the multiplier algebra of $C^*_{\mathrm{u}}(G)$. There is a *universal representation* $u_G : G \to \mathscr{U}(\mathscr{M}(C^*_{\mathrm{u}}(G)))$ defined on elements of $C_c(G)$ by

$$(u_G(t)f)(s) = f(t^{-1}s), \quad (fu_G(t))(s) = f(st^{-1})\Delta(t^{-1}). \tag{2.2}$$

The following lemma is straightforward to check directly; it is essentially the same as [24, Lemma 3.1].

Lemma 2.8 *Let E be a subspace of $B(G)$, and $I := \{a \in C^*_{\mathrm{u}}(G) \mid \langle a, \phi \rangle = 0\ \forall \phi \in E\}$ be its pre-annihilator. Then the following are equivalent:*

1. *E is translation invariant;*
2. *I is invariant under left and right multiplication by the image of G under u_G;*
3. *I is an ideal in $C^*_{\mathrm{u}}(G)$.*

Here then is the definition of Kaliszewski-Landstad-Quigg.

Definition 2.9 Let E be a translation invariant subspace of $B(G)$. Let I_E be the pre-annihilator of E, which is an ideal by Lemma 2.8. Define a C^*-algebra

$$C^*_{E,KLQ}(G) := C^*_{\mathrm{u}}(G)/I_E.$$

This construction actually gives rise to all exotic group algebras (and indeed to all quotients of $C^*_{\mathrm{u}}(G)$ if we do not impose further restrictions on E). To make this statement precise, recall that for us a group C^*-algebra is essentially the same thing as a C^*-algebra norm on $C_c(G)$ that dominates the reduced norm, and is dominated by the universal norm. If $\|\cdot\|$ is such a norm, write $E_{\|\cdot\|}$ for the elements of $B(G)$ that are continuous for that norm; conversely, if E is a translation invariant subspace of $B(G)$, write $\|\cdot\|_{E,KLQ}$ for the norm on $C^*_{E,KLQ}(G)$ restricted to $C_c(G)$. Let $B_{\mathrm{r}}(G)$ denote the elements of $B(G)$ that are continuous for the reduced norm, so $B_{\mathrm{r}}(G)$ identifies canonically with $C^*_{\mathrm{r}}(G)'$. It also coincides with the weak* closure of $B_c(G) := C_c(G) \cap B(G)$ in $B(G)$ by [24, Lemma 3.9].

Proposition 2.10 *The assignments*

$$E \mapsto \|\cdot\|_{E,KLQ}, \quad \|\cdot\| \mapsto E_{\|\cdot\|}$$

define mutually inverse bijections between the set of weak-closed, proper, translation invariant subspaces of $B(G)$ that strictly contain $B_{\mathrm{r}}(G)$, and the set of exotic group algebra norms on $C_c(G)$.*

Proof Recall first that if X is any Banach space, then taking annihilators and pre-annihilators gives a bijective correspondence between closed subspaces of X and weak*-closed subspaces of the dual X'. Specialising this to $X = C^*_{\mathrm{u}}(G)$ and using Lemma 2.8 gives a bijective correspondence between translation invariant weak*-closed subspaces of $X' = B(G)$ and closed ideals of $C^*_{\mathrm{u}}(G)$. Specialising yet further gives a bijective correspondence between the weak*-closed, proper, translation invariant subspaces of $B(G)$ that strictly contain $B_{\mathrm{r}}(G)$, and the closed nonzero ideals in $C^*_{\mathrm{u}}(G)$ that are strictly contained in the kernel of the canonical quotient map $C^*_{\mathrm{u}}(G) \to C^*_{\mathrm{r}}(G)$. The latter are clearly in bijective correspondence with C^*-semi-norms on $C^*_{\mathrm{u}}(G)$ that restrict to exotic group algebra norms on $C_c(G)$ by associating an ideal to the corresponding quotient norm and vice versa. Following these correspondences through gives the result. □

It is worth noticing that by [24, Lemma 3.14] the condition $B_{\mathrm{r}}(G) \subseteq E$ is automatic, if E is a non-zero weak* closed translation invariant *ideal* in $B(G)$.

For a translation invariant space D of functions on G containing $B_c(G) = B(G) \cap C_c(G)$ it is asserted in [24, Lemma 3.5] that $C^*_{D,BG}(G) \cong C^*_{E,KLQ}(G)$ if we define $E := D \cap B(G)$. However there seems to be a gap in the proof of [24, Lemma 3.5] and it is not clear to us whether the conclusion of that lemma holds. So our next goal is to clarify the relationship between the Brown-Guentner and Kaliszewski-Landstad-Quigg constructions. We denote by $P(G)$ the collection of continuous positive type

functions on G, which is a cone in $B(G)$. The following proposition is closely related to [36, Propositions 4.1 and 4.3].

Proposition 2.11 *Say D is a translation invariant subspace of the vector space of complex-valued functions on G. Then $E := span(D \cap P(G))$ is a translation invariant subspace of $B(G)$. Moreover, the identity map on $C_c(G)$ extends to an isomorphism $C^*_{D,BG}(G) \cong C^*_{E,KLQ}(G)$.*

Proof As $P(G)$ is contained in $B(G)$, it is clear that E is a subspace of $B(G)$; we must show that it is translation invariant. We will focus on the left action; the case of the right action is similar.

For a function $\phi : G \to \mathbb{C}$ and $s \in G$, write ${}_s\phi$ for the left-translate defined by ${}_s\phi : t \mapsto \phi(s^{-1}t)$, and $\phi_s : t \mapsto \phi(ts)$ for the right translate. It suffices to show that if $\phi \in P(G) \cap D$ and $s \in G$, then ${}_s\phi$ is in E. As ϕ is in $P(G)$, we may write

$$\phi(t) = \langle \xi, u_t\xi \rangle$$

for some representation $u : G \to \mathscr{U}(\mathscr{H})$ and $\xi \in \mathscr{H}$. It follows that

$${}_s\phi(t) = \langle \xi, u_{s^{-1}t}\xi \rangle = \langle u_s\xi, u_t\xi \rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \langle u_s\xi + i^k\xi, u_t(u_s\xi + i^k\xi) \rangle.$$

Write $\phi_k(t) = \langle u_s\xi + i^k\xi, u_t(u_s\xi + i^k\xi) \rangle$; as each ϕ_k is in $P(G)$, it suffices to show that each is in D. Note, however, that

$$\begin{aligned} \phi_k(t) &= \langle u_s\xi, u_t u_s\xi \rangle + i^k \langle u_s\xi, u_t\xi \rangle + i^{-k} \langle \xi, u_t u_s\xi \rangle + \langle \xi, u_t\xi \rangle \\ &= {}_s\phi_s(t) + i^k\, {}_s\phi(t) + i^{-k}\phi_s(t) + \phi(t), \end{aligned}$$

and so ϕ_k is in D because D is translation invariant.

To see that the identity map on $C_c(G)$ induces an isomorphism $C^*_{D,BG}(G) \cong C^*_{E,KLQ}(G)$, it suffices to check that

$$I_E = \bigcap \{\text{kernel}(u : C^*_{\mathrm{u}}(G) \to \mathscr{L}(\mathscr{H})) \mid u : G \to \mathscr{U}(\mathscr{H}) \text{ a } D\text{-representation}\}$$

Say first $a \in I_E$, and let $u : G \to \mathscr{U}(\mathscr{H})$ be a D-representation with associated dense subspace $\mathscr{H}_0$ giving rise to a dense set of matrix coefficients in D. Then for any $\xi \in \mathscr{H}_0$, if $\phi(s) = \langle \xi, u_s\xi \rangle$ then ϕ is in $D \cap P(G) \subseteq E$ and we have that

$$\langle \xi, u(a)\xi \rangle = \langle a, \phi \rangle = 0.$$

As $\mathscr{H}_0$ is dense, this forces $\langle \eta, u(a)\eta \rangle = 0$ for all $\eta \in \mathscr{H}$, and thus a to be in the kernel of (the integrated form of) u. Hence

$$I_E \subseteq \bigcap \{\text{kernel}(u : C^*_{\mathrm{u}}(G) \to \mathscr{L}(\mathscr{H})) \mid u : G \to \mathscr{U}(\mathscr{H}) \text{ a } D\text{-representation}\}.$$

For the converse inclusion, say $a \in C^*_{\mathrm{u}}(G) \setminus I_E$. Then there is an element ϕ of E such that $\langle a, \phi \rangle \neq 0$. As E is spanned by $D \cap P(G)$, we may assume moreover that ϕ is in $D \cap P(G)$. Let $(u, \mathscr{H}, \xi)$ be the GNS triple associated to ϕ, and note that $u : G \to \mathscr{U}(\mathscr{H})$ is a D-representation: indeed, since D is translation invariant we may take $\mathscr{H}_0 = \mathrm{span}\{u_s\xi \mid s \in G\}$. Hence $\langle \xi, u(a)\xi \rangle = \langle a, \phi \rangle$, which is non-zero, and thus $a \notin \mathrm{kernel}(u : C^*_{\mathrm{u}}(G) \to \mathscr{L}(\mathscr{H}))$. □

Corollary 2.12 *Let $E \subseteq B(G)$ be a translation invariant subspace of $B(G)$. Then the following are equivalent:*

1. *$E_0 = \mathrm{span}(E \cap P(G))$ is weak* dense in E.*
2. *The identity map on $C_c(G)$ extends to an isomorphism $C^*_{E,BG}(G) \cong C^*_{E,KLQ}(G)$.*

Proof It follows from the definition of KLQ-group algebras together with Proposition 2.10 that $C^*_{E_0,KLQ}(G) = C^*_{E,KLQ}(G)$ if and only if the weak* closures of E_0 and E coincide. Combining this with Proposition 2.11 gives the equivalence of (1) and (2). □

At the time of writing we do not know whether $\mathrm{span}(E \cap P(G))$ is weak* dense in E for all translation invariant subspaces of $B(G)$, but it seems likely that there are counter examples. The following lemma gives some results:

Lemma 2.13 *Suppose $E \subseteq B(G)$ is a translation invariant subspace which satisfies one of the following conditions:*

1. *E is closed with respect to the supremum-norm on $B(G) \subseteq C_b(G)$.*
2. *E is closed in the norm on $B(G)$ coming from identifying $B(G)$ with the dual space $C^*_{\mathrm{u}}(G)'$.*
3. *E is weak* closed.*

Then the identity map on $C_c(G)$ extends to an isomorphism $C^*_{E,BG}(G) = C^*_{E,KLQ}(G)$.

Proof We show that in all three cases every element in E can be written as a linear combination of positive elements in E. Having done this, all three cases then follow from Proposition 2.11. For this let $s \mapsto \phi(s) = \langle \xi, u_s\eta \rangle$ be a nonzero element of E for some unitary representation $u : G \to \mathscr{U}(\mathscr{H})$. By passing to $\mathscr{H}_0 = \overline{\mathrm{span}}(u(G)\xi) \cap \overline{\mathrm{span}}(u(G)\eta)$ and the images of ξ, η under the orthogonal projections to $\mathscr{H}_0$, if necessary, we may assume without loss of generality that both vectors ξ, η are cyclic vectors for u. Approximating ξ by elements in $\mathrm{span}(u(G)\eta)$ and observing that $(s \mapsto \langle \xi, u_s\eta' \rangle) \in E$ for any $\eta' \in \mathrm{span}(u(G)\eta)$ by translation invariance of E, it follows from any of the conditions (1), (2), (3) that $(s \mapsto \langle \xi, u_s\xi \rangle) \in E$, and a similar argument gives $(s \mapsto \langle \eta, u_s\eta \rangle) \in E$. But then every summand in the polarisation identity

$$\langle \xi, u_s\eta \rangle = \frac{1}{4} \sum_{k=0}^{3} i^k \langle \xi + i^k\eta, u_s(\xi + i^k\eta) \rangle$$

lies in E. This finishes the proof. □

Example 2.14 The above lemma applies to $E_0 := C_0(G) \cap B(G)$, which is closed in $B(G)$ under $\|\cdot\|_\infty$. We do not know whether the conclusion of the above lemma applies to $D_p := L^p(G) \cap B(G)$, so we do not know whether $C^*_p(G) = C^*_{D_p,BG}(G)$ equals $C^*_{D_p,KLQ}(G)$. However, if we replace D_p by $E_p := \operatorname{span}(L^p(G) \cap P(G))$ we get $C^*_p(G) = C^*_{E_p,KLQ}(G)$.

Convention 2.15 In what follows we shall often use the notation "$C^*_E(G)$" for the *KLQ group algebra* attached to E. Recall that we have $C^*_E(G) = C^*_{\overline{E}}(G)$ if $\overline{E}$ denotes the weak* closure of E in $B(G)$. We shall be careful to write $C^*_{E,BG}(G)$ whenever we want to talk about the BG-group algebra attached to E in cases where it does not obviously coincide with $C^*_E(G)$ by any of the above results.

Remark 2.16 It will be relevant to us that algebra properties of E are reflected in *co*algebra properties of $C^*_E(G)$. The basic result in this direction is due to Kaliszewski, Landstad and Quigg [24, Corollary 3.13 and Proposition 3.16]. To describe it, let $u_G : G \to \mathcal{U}(\mathcal{M}(C^*_{\mathrm{u}}(G)))$ be the universal representation of G as in line (2.2) above. Define the *comultiplication* homomorphism

$$\delta : C^*_{\mathrm{u}}(G) \to \mathcal{M}(C^*_{\mathrm{u}}(G) \otimes C^*_{\mathrm{u}}(G))$$

to be the integrated form of the diagonal representation $s \mapsto u_G(s) \otimes u_G(s)$, which exists by Remark 2.2. Consider the following diagram, where the horizontal arrows are induced by the canonical quotients:

$$\begin{tikzcd}
C^*_{\mathrm{u}}(G) \arrow[r] \arrow[d, "\delta"] & C^*_E(G) \arrow[r, equal] \arrow[d, dashed] & C^*_E(G) \arrow[d, dashed] \\
\mathcal{M}(C^*_{\mathrm{u}}(G) \otimes C^*_{\mathrm{u}}(G)) \arrow[r] & \mathcal{M}(C^*_E(G) \otimes C^*_{\mathrm{u}}(G)) \arrow[r] & \mathcal{M}(C^*_E(G) \otimes C^*_E(G)).
\end{tikzcd}$$

Then E is a subalgebra of $B(G)$ if and only the rightmost dashed arrow can be filled in; and E is an ideal in $B(G)$ if and only if the central dashed arrow can be filled in.

We close this section with three theorems which show that exotic group C^*-algebras allow new characterisations of classical notions from non-abelian harmonic analysis. For discrete groups, these results can be found in [4, Sections 2 and 3], although (1) if and only if (2) from Theorem 2.17 is much older, and due to Hulanicki [21]. The general cases can be proved by slight elaborations of the arguments given there: see also [22] for the result on the Haagerup property in the general case.

Theorem 2.17 *The following are equivalent:*

1. *G is amenable;*
2. *$C_c(G) \cap B(G)$ is weak* dense in $B(G)$, i.e. $C^*_{\mathrm{u}}(G) = C^*_{\mathrm{r}}(G)$;*
3. *$E_p = \operatorname{span}(L^p(G) \cap P(G))$ is weak* dense in $B(G)$ for some $p < \infty$, i.e. $C^*_{\mathrm{u}}(G) = C^*_p(G)$.*

Theorem 2.18 *The following are equivalent:*

1. *G has the Haagerup approximation property;*
2. *$E_0 := B(G) \cap C_0(G)$ is weak* dense in $B(G)$, i.e. $C^*_{\mathrm{u}}(G) = C^*_{E_0}(G)$.*

The proof can be done as the proof of Corollary 3.4 in [4], where a similar result is shown for discrete G and $C^*_{C_0(G),BG}(G)$. By Example 2.14 we know that this algebra coincides with $C^*_{E_0}(G)$. The third theorem is about property (T):

Theorem 2.19 *The following are equivalent:*

1. *G has property (T);*
2. *If E is a translation invariant ideal of $B(G)$ such that $E_0 := \mathrm{span}(E \cap P(G))$ is weak* dense in $B(G)$, then $E = B(G)$;*
3. *If E is a translation invariant ideal of $B(G)$ such that $C^*_{\mathrm{u}}(G) = C^*_{E,BG}(G)$, then $E = B(G)$.*

Proof The equivalence between (2) and (3) follows from the above discussions, since $C^*_{E,BG}(G) = C^*_{E_0}(G) = C^*_{\overline{E_0}}(G)$ equals $C^*_{\mathrm{u}}(G)$ if and only if $\overline{E_0} = B(G)$, where $\overline{E_0}$ denotes the weak* closure of E_0.

An analogue of the equivalence between (1) and (3) has been shown for discrete G in [4, Proposition 3.6], but with $B(G)$ replaced by $\ell^\infty(G)$. The proof of the general case follows along similar lines: Assume that G has property (T) and let E be a translation invariant ideal of $B(G)$ such that $C^*_{E,BG}(G) = C^*_{\mathrm{u}}(G)$. Then there exists a faithful E-representation $u : G \to \mathcal{U}(\mathcal{H})$, e.g., take the direct sum of all GNS-representations attached to elements in $E \cap P(G)$. Then 1_G is weakly contained in u, and therefore, by property (T), 1_G is a subrepresentation of u. Hence there exists a unit vector $\xi \in \mathcal{H}$ such that $u_s\xi = \xi$ for all $s \in G$. Since u is an E-representation, there exists a sequence (ξ_n) of unit vectors in $\mathcal{H}$ which converges to ξ and such that $s \mapsto \phi_n(s) = \langle \xi_n, u_s\xi_n \rangle$ lies in $E \cap P(G)$ for all $n \in \mathbb{N}$. It follows that $\phi_n \to 1_G$ in the norm topology of $B(G)$. Since $B(G)$ is a Banach algebra, it follows that $E \cap P(G)$ contains an invertible element of $B(G)$. Thus $E = B(G)$.

For the converse direction we can use the same arguments as given in the proof of [4, Proposition 3.6]. $\square$

3 Exotic Crossed Products

More details on the standard material on universal and reduced crossed products in this section can be found in [12], [14, Appendix A], and [35].

Let (A, α) be a G-C^*-algebra, i.e. A is a C^*-algebra equipped with a homomorphism α from G to the *-automorphisms of A such that the map $s \mapsto \alpha_s(a)$ is (norm) continuous for all $a \in A$. The natural class of representations of (A, α) are *covariant pairs*: pairs (π, u) consisting of a *-representation of A and a (unitary) representation

of G on the same Hilbert space $\mathscr{H}$ that satisfy the relation

$$u_s^* \pi(a) u_s = \pi(\alpha_s(a)) \tag{3.1}$$

for all $a \in A$ and $s \in G$.

Let $C_c(G, A)$ denote the space of norm continuous, compactly supported functions from G to A, equipped with the *-algebra operations:

$$f * g(t) := \int_G f(s)\alpha_s(g(s^{-1}t))\, ds \quad \text{and} \quad f^*(t) := \Delta(t)^{-1}\alpha_t(f(t^{-1}))^*.$$

Note that any covariant pair (π, u) integrates to a *-representation $\pi \rtimes u$ of $C_c(G, A)$ via the formula

$$\pi \rtimes u(f) = \int_G \pi(f(s))u_s\, ds.$$

It will be useful for later purposes to note that this generalises to actions on Hilbert modules and multiplier algebras in a natural way. Precisely, if $\mathscr{E}$ is a Hilbert B-module then a covariant pair (π, u) for (A, α) on $\mathscr{E}$ consists of a *-representation $\pi : A \to \mathscr{L}(\mathscr{E})$ from A to the adjointable operators on $\mathscr{E}$, and a (strongly continuous, unitary) representation $u : G \to \mathscr{U}(\mathscr{E})$ from G to the group of unitary operators on $\mathscr{E}$ satisfying the compatibility relationship in line (3.1) above. Regarding a C*-algebra D as a Hilbert module over itself, the isomorphism $\mathscr{M}(D) \cong \mathscr{L}(D)$ shows that covariant representations into multiplier algebras with strongly continuous unitary part are special cases of covariant representations on Hilbert modules.

Analogously to the case of group algebras, the following two completions of $C_c(G, A)$ have been very widely studied.

Definition 3.1 The *universal completion* $A \rtimes_{\alpha,\mathrm{u}} G$ is the completion of $C_c(G, A)$ with respect to the norm

$$\|f\|_{\mathrm{u}} := \sup\{\|\pi \rtimes u(f)\| : (\pi, u) \text{ a covariant pair}\}.$$

Let π be the representation of A on the Hilbert A-module $L^2(G, A)$ defined by the formula $(\pi(a)\xi)(s) = \alpha_{s^{-1}}(a)\xi(s)$. Let $\lambda \otimes 1$ be the representation of G on this Hilbert module defined by $((\lambda \otimes 1)_t\xi)(s) = \xi(t^{-1}s)$. These representations define a covariant pair $(\pi, \lambda \otimes 1)$ on the Hilbert module $L^2(G, A)$.

The *reduced crossed product* $A \rtimes_{\alpha,\mathrm{r}} G$ is, by definition, the completion of $C_c(G, A)$ for the norm

$$\|f\|_{\mathrm{r}} := \|\pi \rtimes (\lambda \otimes 1)(f)\|.$$

The universal crossed product has the universal property that any covariant pair (π, u) with values in a Hilbert module $\mathscr{E}$ (in particular, in a Hilbert space) for (A, α)

integrates to a *-homomorphism

$$\pi \rtimes_{\mathrm{u}} u : A \rtimes_{\alpha,\mathrm{u}} G \to \mathscr{L}(\mathscr{E})$$

to the bounded (adjointable) operators on the Hilbert space (module) and, conversely, that every nondegenerate *-representation σ of $A \rtimes_{\alpha,\mathrm{u}} G$ is the integrated form of some (nondegenerate) covariant representation: There is a universal covariant representation (ι_A, ι_G) of (A, G) into $\mathscr{M}(A \rtimes_{\alpha,\mathrm{u}} G)$ such that $\sigma = \pi \rtimes u$ with $\pi = \sigma \circ \iota_A$ and $u = \sigma \circ \iota_G$. On the level of $C_c(G, A)$ the universal covariant pair is given by the formulas

$$\iota_A(a)f(s) = af(s), \quad \iota_G(t)f(s) = \alpha_t(f(t^{-1}s)). \tag{3.2}$$

Analogously to the group case, an exotic crossed product is roughly a completion of $C_c(G, A)$ for a norm between the maximal and reduced norms. Motivated by both examples and applications, it seems reasonable to ask for some compatibility between such exotic crossed products as A varies. The minimal reasonable requirement here seems to be compatibility with *-homomorphisms (we will discuss some stronger requirements later): to make this precise, note that if $\phi : A \to B$ is a G-equivariant *-homomorphism then the function

$$\phi \rtimes_c G : C_c(G, A) \to C_c(G, B), \quad f \mapsto \phi \circ f \tag{3.3}$$

is a *-homomorphism, and moreover, the assignment $\phi \mapsto \phi \rtimes_c G$ is functorial.

Definition 3.2 A *crossed product* is an assignment to each G-C^*-algebra (A, α) of a completion $A \rtimes_{\alpha,\mu} G$ of $C_c(G, A)$ for a C^*-norm $\|\cdot\|_\mu$ such that:

1. for all $f \in C_c(G, A)$,

$$\|f\|_{\mathrm{u}} \geq \|f\|_\mu \geq \|f\|_{\mathrm{r}};$$

2. for any equivariant *-homomorphism $\phi : (A, \alpha) \to (B, \beta)$ the *-homomorphism $\phi \rtimes_c G$ of line (3.3) extends to a *-homomorphism

$$\phi \rtimes_\mu G : A \rtimes_{\alpha,\mu} G \to B \rtimes_{\beta,\mu} G.$$

A crossed product is *exotic* if the associated norm differs from the maximal and reduced norms (on at least one G-C^*-algebra each).

Thus a crossed product is a functor from the category of G-C^*-algebras and equivariant *-homomorphisms to the category of C^*-algebras and *-homomorphisms that sits between the universal and reduced completions in some sense. One might wonder why we only consider completions of $C_c(G, A)$ by norms $\|\cdot\|_\mu$ which dominate the reduced norm $\|\cdot\|_{\mathrm{r}}$. We come back to this point in Sect. 4.4

There are natural extensions of both the Brown-Guentner and Kaliszewski-Landstad-Quigg exotic group algebra constructions to exotic crossed products. Here is the Brown-Guentner construction.

Definition 3.3 Let $C^*_E(G)$ be a group C^*-algebra as in Convention 2.15, and (A, α) a dynamical system. The *Brown-Guentner crossed product* $A \rtimes_{\alpha,E_{BG}} G$ (for short: BG crossed product) is the completion of $C_c(G, A)$ for the norm

$$\|f\|_{E_{BG}} := \sup\{\|\pi \rtimes u(f)\| \mid (\pi, u) \text{ a covariant pair such that } u \text{ extends to } C^*_E(G)\}.$$

In order to define the Kaliszewski-Landstad-Quigg construction of crossed products, we need a little more notation. Let (ι_A, ι_G) be the universal covariant representation of (A, G) into $\mathcal{M}(A \rtimes_{\alpha,\mathrm{u}} G)$ as in line (3.2) above. Let $u_G: G \to \mathcal{M}(C^*_{\mathrm{u}}(G))$ be the universal representation of G as in line (2.2) above (that is, u_G coincides with $\iota_G : G \to \mathcal{U}\mathcal{M}(\mathbb{C} \rtimes_{\mathrm{u}} G)$ if we identify $C^*_{\mathrm{u}}(G)$ with $\mathbb{C} \rtimes_{\mathrm{u}} G$). Then the maps

$$\iota_A \otimes 1 : A \to \mathcal{M}(A \rtimes_{\alpha,\mathrm{u}} G \otimes C^*_{\mathrm{u}}(G)), \quad \iota_G \otimes u_G : G \to \mathcal{U}\mathcal{M}(A \rtimes_{\alpha,\mathrm{u}} G \otimes C^*_{\mathrm{u}}(G))$$

define a covariant pair for (A, α) (here and throughout the rest of the paper, "$\otimes$" denotes the spatial tensor product of C^*-algebras). The integrated form of this covariant pair (which exists by the universal property of $A \rtimes_{\alpha,\mathrm{u}} G$) is a *-homomorphism

$$\hat{\alpha} : A \rtimes_{\alpha,\mathrm{u}} G \to \mathcal{M}(A \rtimes_{\alpha,\mathrm{u}} G \otimes C^*_{\mathrm{u}}(G)) \tag{3.4}$$

called the *dual coaction* associated to (A, α).

Definition 3.4 Let $C^*_E(G)$ be a group C^*-algebra, and (A, α) a dynamical system. Let $q_E : C^*_{\mathrm{u}}(G) \to C^*_E(G)$ denote the canonical quotient map, and let

$$\mathrm{id} \otimes q_E : \mathcal{M}(A \rtimes_{\alpha,\mathrm{u}} G \otimes C^*_{\mathrm{u}}(G)) \to \mathcal{M}(A \rtimes_{\alpha,\mathrm{u}} G \otimes C^*_E(G))$$

denote the extension of the canonical tensor product *-homomorphism to the multiplier algebras.

The *Kaliszewski-Landstad-Quigg crossed product* $A \rtimes_{\alpha,E_{KLQ}} G$ is the completion of $C_c(G, A)$ for the norm

$$\|f\|_{E_{KLQ}} := \|(\mathrm{id} \otimes q_E) \circ \hat{\alpha}(f)\|.$$

The properties of the BG and KLQ crossed products that we will use are recorded below. We will discuss some more properties of these functors in the next section. Proofs of these results (in a slightly different form) and other basic facts about BG and KLQ crossed products can be found in [3, Appendix A], [24, Section 6], and [6, Section 5].

Proposition 3.5 *Let $C^*_E(G)$ be an exotic group algebra. The following facts hold for the associated BG and KLQ crossed products.*

1. *The BG and KLQ crossed products are both functorial for equivariant *-homomorphisms, and in particular are crossed products in the sense of Definition 3.2.*
2. *The correspondences $E \mapsto \rtimes_{E_{BG}}$ and $\rtimes_{E_{BG}} \mapsto (\mathbb{C} \rtimes_{E_{BG}} G)'$ are mutually inverse bijections between the collection of all BG functors and all weak*-closed translation invariant subspaces of $B(G)$ that contain $B_{\mathrm{r}}(G)$. In particular, $\mathbb{C} \rtimes_{E_{BG}} G$ identifies with $C^*_E(G)$ via a *-isomorphism that extends the identity map on $C_c(G)$.*
3. *The correspondences $E \mapsto \rtimes_{E_{KLQ}}$ and $\rtimes_{E_{KLQ}} \mapsto (\mathbb{C} \rtimes_{E_{KLQ}} G)'$ are mutually inverse bijections between the collection of all KLQ functors and all weak*-closed translation invariant ideals in $B(G)$ that contain $B_{\mathrm{r}}(G)$. In particular, for any group C^*-algebra $C^*_E(G)$, $\mathbb{C} \rtimes_{E_{KLQ}} G$ identifies with $C^*_{\langle E\rangle}(G)$ via a *-isomorphism that extends the identity map on $C_c(G)$, where $\langle E\rangle$ is the weak*-closed ideal in $B(G)$ generated by E.*

In the next section, we will study functorial properties of the BG and KLQ crossed products in much more detail.

We conclude this section with some rather unnatural examples that are useful for constructing crossed products with 'bad' properties. For yet another construction of exotic crossed products, see [9, Section 2.4 and Corollary 4.20].

Example 3.6 Let $\mathscr{S}$ be a collection of G-C^*-algebras. For any G-C^*-algebra (A, α), define a seminorm on $C_c(G, A)$ by

$$\|f\|_{\mathscr{S},0} := \sup\{\|\phi \rtimes_c G(f)\|_{\mathrm{u}} : \phi \in \mathrm{Mor}_G(A, B) \text{ for some } B \in \mathscr{S}\},$$

where $\mathrm{Mor}_G(A, B)$ denotes the set of G-equivariant *-homomorphisms $A \to B$. We then define a norm on $C_c(G, A)$ by

$$\|f\|_{\mathscr{S}} := \max\{\|f\|_{\mathrm{r}}, \|f\|_{\mathscr{S},0}\}.$$

If $A \rtimes_{\alpha,\mathscr{S}} G$ is the associated completion, then the assignment $(A, \alpha) \mapsto A \rtimes_{\alpha,\mathscr{S}} G$ is a crossed product functor (see [9, Lemma 2.5]).

4 Properties of Crossed Products

We start this section by discussing some strong functoriality properties that a crossed product functor can have and give some useful characterisations of these. We then discuss some applications to K-theory computations, duality theory and tensor products. We also discuss in Sect. 4.4 "pseudo crossed products", which are certain quotients of the full crossed product that do not necessarily lie above the reduced crossed product.

Before we start stating the properties of interest, we need to give a brief discussion about crossed products of G-equivariant Hilbert modules and correspondences. If (B, β) is a G-algebra and $\mathscr{E}$ is a Hilbert B-module, then a compatible action of G on $\mathscr{E}$ is a strongly continuous homomorphism $\gamma : G \to \mathrm{Aut}(\mathscr{E})$ such that

$$\gamma_s(xb) = \gamma_s(x)\beta_s(b) \quad \text{and} \quad \langle \gamma_s(x), \gamma_s(y)\rangle = \beta_s(\langle x, y\rangle),$$

for all $x, y \in E, a \in A$ and $s \in G$. If $(A, \alpha) \mapsto A \rtimes_{\alpha,\mu} G$ is an exotic crossed-product functor we may extend this functor to equivariant Hilbert modules as follows: If $(\mathscr{E}, \gamma)$ is a G-equivariant (B, β) Hilbert module as above, then there is a well-known canonical $C_c(G, B)$-valued inner product on $C_c(G, \mathscr{E})$ together with a compatible right module action of $C_c(G, B)$ on $C_c(G, \mathscr{E})$ given by

$$\langle x \,|\, y\rangle_{C_c(G,B)}(t) = \int_G \beta_{s^{-1}}(\langle x(s) \,|\, y(st)\rangle_B \, ds, \quad (x \cdot \varphi)(t) = \int_G x(s)\beta_{s^{-1}}(\varphi(s^{-1}t))\, ds$$

for $x, y \in C_c(G, \mathscr{E})$ and $\varphi \in C_c(G, B)$. If we regard $C_c(G, B)$ a subalgebra of $B \rtimes_{\beta,\mu} G$, we obtain a norm $\|x\|_\mu = \sqrt{\|\langle x \,|\, x\rangle\|_\mu}$ on $C_c(G, \mathscr{E})$. The actions and inner products then extend to the μ-completions so that we obtain a $B \rtimes_{\beta,\mu} G$-Hilbert module $\mathscr{E} \rtimes_{\gamma,\mu} G$.

If (A, α) is another G-algebra, then a *G-equivariant* $(A, \alpha) - (B, \beta)$ *correspondence* is a triple $(\mathscr{E}, \gamma, \phi)$ in which $(\mathscr{E}, \gamma)$ is a G-equivariant (B, β)-Hilbert module and $\phi : A \to \mathscr{L}(\mathscr{E})$ is an $\alpha - \mathrm{Ad}\gamma$-equivariant *-homomorphism (possibly degenerate). There is a category $\mathfrak{Corr}(G)$ in which the objects are G-C*-algebras and the morphisms are equivalence classes of $(A, \alpha) - (B, \beta)$ correspondences, where $(\mathscr{E}, \gamma, \phi) \sim (\mathscr{E}', \gamma', \phi')$ if there is an isomorphism $\phi(A)\mathscr{E} \xrightarrow{\sim} \phi'(A)\mathscr{E}'$ of Hilbert B-modules commuting the left actions of A. Composition of correspondences is given by taking internal tensor products

$$[(\mathscr{E}, \gamma, \phi)][(\mathscr{F}, \nu, \psi)] = [(\mathscr{E} \otimes_B \mathscr{F}, \gamma \otimes \nu, \phi \otimes 1)].$$

We write $\mathfrak{Corr} := \mathfrak{Corr}(\{e\})$ for the correspondence category of the trivial group $\{e\}$. Isomorphisms in the correspondence categories are precisely the Morita equivalences, i.e. correspondences where $\phi : A \to \mathscr{L}(\mathscr{E})$ induces an isomorphism $A \cong \mathscr{K}(\mathscr{E})$. Correspondence categories have been studied extensively in the literature (e.g. see [14]), where usually the homomorphisms $\Phi : A \to \mathscr{L}(\mathscr{E})$ are assumed to be nondegenerate. But notice that every correspondence in our sense is equivalent to a nondegenerate correspondence, so the resulting categories are equivalent.

Definition 4.1 Let $(A, \alpha) \to A \rtimes_{\alpha,\mu} G$ be a crossed-product functor. This functor:

1. *extends to generalised homomorphisms* if for any (possibly degenerate) G-equivariant *-homomorphism $\phi : A \to \mathscr{M}(B)$ there exists a *-homomorphism

$$\phi \rtimes_\mu G : A \rtimes_\mu G \to \mathscr{M}(B \rtimes_\mu G)$$

which is given on the level of functions $f \in C_c(G,A)$ by $f \mapsto \phi \circ f$ in the sense that $\phi \rtimes_\mu G(f)g = (\phi \circ f) * g$ for all $g \in C_c(G,B)$;
2. has the *ideal property* if for every G-invariant closed ideal in a G-algebra A, the inclusion map $\iota : I \hookrightarrow A$ descends to an injective *-homomorphism $\iota \rtimes_\mu G : I \rtimes_\mu G \hookrightarrow A \rtimes_\mu G$;
3. is *strongly Morita compatible* if for every G-equivariant $(A,\alpha) - (B,\beta)$ equivalence bimodule $(\mathscr{E},\gamma)$, the left action of $C_c(G,A)$ on $C_c(G,\mathscr{E})$ given by

$$(\phi \rtimes G(f)x)(t) := \int_G \phi(f(s))\gamma_s(x(s^{-1}t))\,ds$$

extends to an action of $A \rtimes_{\alpha,\mu} G$ on $\mathscr{E} \rtimes_{\gamma,\mu} G$ such that $\mathscr{E} \rtimes_{\gamma,\mu} G$ becomes an $A \rtimes_{\alpha,\mu} G - B \rtimes_{\beta,\mu} G$ equivalence bimodule;
4. is a *correspondence functor* if for every G-equivariant $(A,\alpha) - (B,\beta)$ correspondence $(\mathscr{E},\gamma,\phi)$, the left action of $C_c(G,A)$ on $C_c(G,\mathscr{E})$ above extends to an action $\phi \rtimes_\mu G$ of $A \rtimes_{\alpha,\mu} G$ on $\mathscr{E} \rtimes_{\gamma,\mu} G$ such that $(\mathscr{E} \rtimes_{\gamma,\mu} G, \phi \rtimes_\mu G)$ becomes an $A \rtimes_{\alpha,\mu} G - B \rtimes_{\beta,\mu} G$ correspondence;
5. has the *(full) projection property* if for every G-algebra A and every G-invariant (full) projection $p \in \mathscr{M}(A)$, the inclusion $\iota : pAp \hookrightarrow A$ descends to a faithful homomorphism $\iota \rtimes_\mu G : pAp \rtimes_{\alpha,\mu} G \to A \rtimes_{\alpha,\mu} G$;
6. has the *(full) hereditary-subalgebra property* if for every (full) hereditary G-invariant subalgebra B of A, the inclusion $\iota : B \hookrightarrow A$ descends to a faithful map $\iota \rtimes_\mu G : B \rtimes_{\alpha,\mu} G \to A \rtimes_{\alpha,\mu} G$;
7. has the *cp map property* if for any completely positive and G-equivariant map $\phi : A \to B$ of G-algebras, the map

$$C_c(G,A) \to C_c(G,B), \quad f \mapsto \phi \circ f$$

extends to a completely positive map from $A \rtimes_{\alpha,\mu} G$ to $B \rtimes_{\beta,\mu} G$.

Remark 4.2 Of course, it follows from the requirements for a correspondence functor $\rtimes_\mu$ that it extends to a functor $\rtimes_\mu : \mathfrak{Corr}(G) \to \mathfrak{Corr}$, and similarly a functor with the cp map property extends to a functor from the category of G-C^*-algebras and equivariant completely positive maps to the category of C^*-algebras and completely positive maps.

We now record the relationships between these various properties. Proposition 4.3 is proved in [9, Section 3], and Theorems 4.4 and 4.5 in [9, Section 4].

Proposition 4.3 *The following are equivalent for a crossed product functor:*

1. *the ideal property;*
2. *extension to generalised morphisms.*

Theorem 4.4 *The following are equivalent for a crossed product functor:*

1. *strong Morita compatibility;*
2. *the full hereditary subalgebra property;*
3. *the full projection property.*

Theorem 4.5 *The following are equivalent for a crossed product functor:*

1. *being a correspondence functor;*
2. *the projection property;*
3. *the hereditary subalgebra property;*
4. *the cp map property;*
5. *having the properties in Theorem 4.4 and Proposition 4.3.*

The properties of BG and KLQ functors listed below are proved in [9, Sections 4 and 5].

Example 4.6 The BG crossed product associated to a group algebra $C^*_E(G)$ always has the ideal property. It is strongly Morita compatible if and only if it is a correspondence functor, if and only if the canonical quotient map $C^*_{\mathrm{u}}(G) \twoheadrightarrow C^*_E(G)$ is an isomorphism.

Example 4.7 KLQ crossed products are always correspondence functors, and thus have all the properties considered above. It is conceivable that all correspondence functors are KLQ functors. This seems unlikely, however, partly as there is a construction of correspondence functors that are not obviously KLQ functors: see [9, Section 2.4 and Corollary 4.20].

On the other hand, the ideal property does not always hold, as the following example shows.

Example 4.8 Let $\mathscr{S}$ consist of $C_0(0,1]$ equipped with the trivial action, and apply the construction of Example 3.6 for this $\mathscr{S}$. Then the inclusion $C_0(0,1] \to C[0,1]$ of trivial G-algebras does not induce an injection on crossed products for any non-amenable G.

We do not, however, know if there are strongly Morita compatible crossed products without the ideal property (we guess the answer is yes, by an elaboration of the above, but the details are currently elusive).

The following theorem is one of the main applications [9, Section 6] of our correspondence functor machinery; it can be regarded as another good functoriality property of correspondence functors.

Theorem 4.9 *Say $(A,\alpha) \mapsto A \rtimes_{\alpha,\mu} G$ is a correspondence functor. Then there exists a* descent functor $\rtimes_\mu : KK^G \to KK$ *that agrees with $\rtimes_\mu$ on objects, and on morphisms coming from equivariant *-homomorphisms.*

Moreover, if G is K-amenable, then the canonical quotients

$$A \rtimes_{\alpha,\mathrm{u}} G \twoheadrightarrow A \rtimes_{\alpha,\mu} G \twoheadrightarrow A \rtimes_{\alpha,\mathrm{r}} G$$

are KK-equivalences.

4.1 K-Theory of Exotic Group Algebras

Corollary 4.10 *Let G be a K-amenable group, and let E be a translation invariant algebraic ideal in $B(G)$ which contains $B_c(G) = B(G) \cap C_c(G)$. Then the canonical quotient maps*

$$C^*_{\mathrm{u}}(G) \twoheadrightarrow C^*_{E,BG}(G) \twoheadrightarrow C^*_{\mathrm{r}}(G), \quad C^*_{\mathrm{u}}(G) \twoheadrightarrow C^*_{E,KLQ}(G) \twoheadrightarrow C^*_{\mathrm{r}}(G)$$

are all KK-equivalences.

Proof Using Theorem 4.9, it suffices to show that $C^*_{E,BG}(G)$ and $C^*_{E,KLQ}(G)$ are of the form $\mathbb{C} \rtimes_\mu G$ for some correspondence functor $\rtimes_\mu$.

For $C^*_{E,KLQ}(G)$, let $\overline{E}$ be the weak* closure of E in $B(G)$, which is an ideal by weak* continuity of multiplication on $B(G)$. Let $\rtimes_\mu$ be the KLQ crossed product functor associated to $C^*_{\overline{E},KLQ}(G)$. Proposition 3.5 part (3) implies that $\mathbb{C} \rtimes_\mu G$ identifies with $C^*_{\overline{E},KLQ}(G)$; however $C^*_{\overline{E},KLQ}(G)$ is clearly the same as $C^*_{E,KLQ}(G)$ by definition of KLQ group algebras.

For $C^*_{E,BG}(G)$, let $\widetilde{E}$ be the span of $P(G) \cap E$, and let F be the weak*-closure of $\widetilde{E}$. Then $C^*_{E,BG}(G) = C^*_{F,KLQ}(G)$ by Corollary 2.12. As $P(G)$ is closed under products in $B(G)$ and every element in $B(G)$ is a linear combination of elements in $P(G)$, $\widetilde{E}$ is an ideal in $B(G)$ and F is a weak*-closed ideal. Thus the result follows from the result for KLQ group algebras. □

Example 4.11 Say $G = F_2$ or $G = SL(2,\mathbb{R})$. Then G is K-amenable, so Corollary 4.10 implies that the uncountably many exotic group C^*-algebras $C^*_p(G)$ from Theorem 2.6 are all KK-equivalent.

Example 4.12 It is tempting from the above to guess that if G is K-amenable, then *all* exotic group algebras (or even crossed products) have the same K-theory. This is false: in fact any non-amenable group admits an exotic group algebra such that the canonical quotient $C^*_E(G) \to C^*_{\mathrm{r}}(G)$ does not even induce an isomorphism on K-theory. This can be achieved by setting $E = B_{\mathrm{r}}(G) \oplus \mathbb{C}1$, for example.

Compare also Remark 4.17 in this regard, which implies that Corollary 4.10 is in some sense the best possible result that can be deduced about K-theory of group algebras using our correspondence functor machinery.

4.2 Duality

Definition 4.13 A crossed product functor $\rtimes_\mu$ is a *duality functor* if there is a *-homomorphism $\hat{\alpha}_\mu$ making the diagram below commute

$$\begin{tikzcd} A \rtimes_{\alpha,\mathrm{u}} G \arrow[r, "\hat{\alpha}"] \arrow[d] & \mathcal{M}(A \rtimes_{\alpha,\mathrm{u}} G \otimes C^*_{\mathrm{u}}(G)) \arrow[d] \\ A \rtimes_{\alpha,\mu} G \arrow[r, dashed, "\hat{\alpha}_\mu"] & \mathcal{M}(A \rtimes_{\alpha,\mu} G \otimes C^*_{\mathrm{u}}(G)) \end{tikzcd},$$

where $\hat{\alpha}$ is the dual coaction of line (3.4) above, and the vertical maps are the canonical quotients.

For the proof of the following theorem see [9, Section 6].

Theorem 4.14 *Correspondence functors are duality functors.*

Remark 4.15 Let $C^*_E(G)$ be a group algebra as in Convention 2.15. It is not difficult to see that the associated BG crossed product is a duality functor if and only if E is an ideal in $B(G)$. In particular, it follows from Example 4.6 that Theorem 4.14 is not optimal.

Remark 4.16 Note that every duality functor $\rtimes_\mu$ admits a version of Imai-Takai duality: The homomorphism $\widehat{\alpha}_\mu$ is a coaction and there is a canonical isomorphism $A \rtimes_{\alpha,\mu} G \rtimes_{\widehat{\alpha}_\mu} \widehat{G} \cong A \otimes \mathcal{K}(L^2(G))$. We refer to [9, Section 6] for more details on this and to [8, 26] for results which show how duality techniques combined with Theorem 4.14 can be efficiently used to extend correspondence crossed-product functors to other categories like Fell-bundles over G or partial G-actions.

Remark 4.17 If $\rtimes_\mu$ is a duality functor and E is the dual space of $\mathbb{C} \rtimes_\mu G$, thought of as a subspace of $B(G)$, then $\mathbb{C} \rtimes_\mu G \cong C^*_E(G)$ carries a coaction of G and hence E is an ideal of $B(G)$ by Remark 2.16. Combined with Proposition 3.5 part (3) and Example 4.7, it follows that an exotic group algebra $C^*_E(G)$ is of the form $\mathbb{C} \rtimes_\mu G$ for a correspondence functor $\rtimes_\mu$ if and only if the dual space of $C^*_E(G)$ is an ideal in $B(G)$.

In particular, the result of Corollary 4.10 is in some sense the optimal application of Theorem 4.9 to computing the K-theory of exotic group algebras.

4.3 Tensor Products

As an example of an application of Theorem 4.5 that does not appear in our paper [9], here we discuss the relationship of crossed products and spatial tensor products. In the next section, we will apply this to discuss the relationship between general crossed products and the reduced crossed product.

Let $\rtimes_\mu$ be a crossed product for G, which is functorial for generalised morphisms. Let (A, α) be a G-C^*-algebra, and let (B, id) be a trivial G-C^*-algebra. Then there is an equivariant *-homomorphism $A \to \mathscr{M}(A \otimes B)$ defined by $a \mapsto a \otimes 1$, and functoriality for generalised morphisms implies that this integrates to a *-homomorphism

$$A \rtimes_{\alpha,\mu} G \to \mathscr{M}((A \otimes B) \rtimes_{\alpha\otimes\mathrm{id},\mu} G). \tag{4.1}$$

Assume now moreover that either B or $A \rtimes_{\alpha,\mu} G$ is nuclear. Then as the natural *-homomorphism $B \to \mathscr{M}((A \otimes B) \rtimes_{\alpha\otimes\mathrm{id},\mu} G)$ commutes with the image of the *-homomorphism in line (4.1), our nuclearity assumptions give a *-homomorphism

$$(A \rtimes_{\alpha,\mu} G) \otimes B \to \mathscr{M}((A \otimes B) \rtimes_{\alpha\otimes\mathrm{id},\mu} G);$$

checking on generators, it is not difficult to see that the image of this *-homomorphism is in fact in $(A \otimes B) \rtimes_{\alpha\otimes\mathrm{id},\mu} G$, and thus we have a canonical *-homomorphism

$$(A \rtimes_{\alpha,\mu} G) \otimes B \to (A \otimes B) \rtimes_{\alpha\otimes\mathrm{id},\mu} G. \tag{4.2}$$

Definition 4.18 Let $(A, \alpha) \mapsto A \rtimes_{\alpha,\mu} G$ be a crossed product functor which is functorial for generalised morphisms. The functor has the *tensor product property* if whenever B is a trivial G-C^*-algebra and (A, α) is a G-C^*-algebra such that one of B or $A \rtimes_{\alpha,\mu} G$ is nuclear, the map in line (4.2) above is an isomorphism.

Since the map in line (4.2) is always surjective the tensor product property is equivalent to the injectivity of (4.2).

Example 4.19 The BG functor associated to a group algebra $C^*_E(G)$ has the tensor product property. Indeed, we have already noted that BG functors are functorial for generalised morphisms. It suffices to prove that there is a faithful representation π of $(A \rtimes_{\alpha,E_{BG}} G) \otimes B$ that extends to $(A \otimes B) \rtimes_{\alpha\otimes\mathrm{id},E_{BG}} G$. Now, by definition of the BG crossed product and the spatial tensor product, we may take a faithful π of the form $(\sigma \rtimes u) \otimes \rho$ for some covariant pair (σ, u) for (A, α) with u an E-representation, and ρ a representation of B. The desired extension is then the integrated form of $(\sigma \otimes \rho, u \otimes \mathrm{id})$, which is a covariant pair for $A \otimes B$ with $u \otimes \mathrm{id}$ an E-representation.

Remark 4.20 If B is unital, then the discussion above makes sense without assuming that $\rtimes_\mu$ is functorial for generalised morphisms, and thus the map in line (4.2) makes sense for any unital nuclear B. At this level of generality, the map in line (4.2) can certainly fail to be an isomorphism, even for $B = M_2(\mathbb{C})$: use the construction in Example 3.6 with $\mathscr{S}$ consisting of $M_2(\mathbb{C})$ equipped with the trivial representation [9, Example 2.6].

Remark 4.21 On the other hand, the map in line (4.2) is always an isomorphism for unital and commutative B, and always an isomorphism for B commutative if $\rtimes_\mu$ is functorial for generalised homomorphisms [9, Remark 3.7 and Lemma 3.6] (it

is false for general commutative C^*-algebras without assuming the ideal property: compare Example 4.8 above).

Proposition 4.22 *Correspondence functors have the tensor product property.*

Proof We must show that the map in line (4.2) is injective, so assume for contradiction that x is a non-trivial element of the kernel of the map above. Then there is a state ϕ on B such that the slice map

$$1_{A\rtimes_{\alpha,\mu}G}\otimes\phi:(A\rtimes_{\alpha,\mu}G)\otimes B\to A\rtimes_{\alpha,\mu}G$$

is non-zero on x. As the slice map $1_A\otimes\phi:A\otimes B\to A$ is an equivariant completely positive map, the cp map property implies that it integrates to a map $(1_A\otimes\phi)\rtimes_\mu G$ on the μ-crossed products. This gives rise to a diagram

$$\begin{tikzcd}
(A\rtimes_{\alpha,\mu}G)\otimes B \arrow[r] \arrow[d, "1_{A\rtimes_{\alpha,\mu}G}\otimes\phi"] & (A\otimes B)\rtimes_{\alpha\otimes\mathrm{id},\mu}G \arrow[d, "(1_A\otimes\phi)\rtimes_\mu G"] \\
A\rtimes_{\alpha,\mu}G \arrow[r, equal] & A\rtimes_{\alpha,\mu}G\,,
\end{tikzcd}$$

which commutes by checking on the dense subalgebra $C_c(G,A)\otimes_{\mathrm{alg}}B$. As the 'right-down' composition sends x to zero, and the 'down-right' composition does not, we have our contradiction. □

4.4 Crossed Products and the Reduced Group C^*-Algebra

Throughout this paper, we only consider exotic group algebras and crossed products that dominate the reduced group algebra. Here we discuss the sort of degeneracy that can occur if one does not do this; some of the ideas underlying this section were pointed out to us by Joachim Cuntz.

For the purposes of this subsection only, by a *pseudo-crossed product* functor we mean a functor that satisfies all of the conditions of Definition 3.2 except possibly that it does not dominate the reduced crossed product norm, and that possibly the norm $\|\cdot\|_\mu$ is only a semi-norm on $C_c(G,A)$ for some C^*-algebras (A,α). Similarly, a *pseudo group algebra* is a C^*-algebra completion of $C_c(G)$ that satisfies the conditions of Definition 2.3 except that possibly the norm $\|\cdot\|_\mu$ does not dominate the reduced norm, and is only a semi-norm on $C_c(G)$. Note that the definitions of the BG and KLQ crossed products still make sense if we allow pseudo group C^*-algebras as the input, but then give rise to pseudo crossed products. Note moreover that the definitions of functoriality for generalised morphisms, and of the tensor product property still make sense.

Lemma 4.23 *Say $\rtimes_\mu$ is a pseudo-crossed product functor that is functorial for generalised morphisms, and such that the crossed product $C_0(G) \rtimes_{\tau,\mu} G$ (where τ is the left translation action) is non-zero. Then $\mathbb{C} \rtimes_\mu G$ dominates the reduced group C^*-algebra.*

If moreover $\rtimes_\mu$ has the tensor product property, then $\rtimes_\mu$ dominates $\rtimes_{\mathrm{r}}$.

Proof From generalised functoriality, the unit inclusion $\mathbb{C} \to \mathscr{M}(C_0(G))$ induces a *-homomorphism $\mathbb{C} \rtimes_\mu G \to \mathscr{M}(C_0(G) \rtimes_{\tau,\mu} G)$. If $C_0(G) \rtimes_\mu G$ is non-zero, then it canonically identifies with $\mathscr{K}(L^2(G))$ (the compact operators on $L^2(G)$), as this is true for $\rtimes_{\mathrm{u}}$, and as $\mathscr{K}(L^2(G))$ is simple. Thus $\mathscr{M}(C_0(G) \rtimes_{\tau,\mu} G)$ identifies canonically with $\mathscr{L}(L^2(G))$. However, it is easy to see that the composition

$$\mathbb{C} \rtimes_\mu G \to \mathscr{M}(C_0(G) \rtimes_{\tau,\mu} G) \cong \mathscr{L}(L^2(G))$$

is the integrated form of the regular representation.

Assume now that in addition $\rtimes_\mu$ has the tensor product property. Then, for any G-C^*-algebra the canonical inclusion $A \to \mathscr{M}(A \otimes C_0(G))$, $a \mapsto a \otimes 1$ gives rise, by generalised functoriality, to a *-homomorphism

$$A \rtimes_{\alpha,\mu} G \to \mathscr{M}((A \otimes C_0(G)) \rtimes_{\alpha\otimes\tau,\mu} G).$$

On the other hand, using the $\alpha \otimes \tau - \mathrm{id} \otimes \tau$-equivariant *-automorphism ϕ of $A \otimes C_0(G) \cong C_0(G, A)$ given by $\phi(f)(g) = \alpha_{g^{-1}}(f(g))$ for $f \in C_0(G, A)$, the tensor product property and the fact (as above) that $C_0(G) \rtimes_{\tau,\mu} G = \mathscr{K}(L^2(G))$ gives

$$\begin{aligned}
\mathscr{M}((A \otimes C_0(G)) \rtimes_{\alpha\otimes\tau,\mu} G) &\cong \mathscr{M}((A \otimes C_0(G)) \rtimes_{\mathrm{id}\otimes\tau,\mu} G) \\
&\cong \mathscr{M}(A \otimes (C_0(G) \rtimes_{\tau,\mu} G)) \\
&\cong \mathscr{M}(A \otimes \mathscr{K}(L^2(G))).
\end{aligned}$$

It is not difficult to check that the composed map

$$A \rtimes_{\alpha,\mu} G \to \mathscr{M}(A \otimes \mathscr{K}(L^2(G)))$$

is induced by an integrated form of the regular representation. □

A similar argument can be used for functors without the ideal property whenever G admits a unital C^*-dynamical system (A, α) such that the maximal crossed product $A \rtimes_{\alpha,\mathrm{u}} G$ is simple. Indeed, say then $\rtimes_\mu$ is a pseudo-crossed product. If $A \rtimes_{\mu,\alpha} G$ is non-zero, then the canonical quotient map $A \rtimes_{\alpha,\mathrm{u}} G \twoheadrightarrow A \rtimes_{\alpha,\mu} G$ is an isomorphism (and similarly for the reduced crossed product); hence the unit

inclusion $\mathbb{C} \to A$ induces a commutative diagram

$$\begin{tikzcd}
\mathbb{C} \rtimes_{\mathrm{r}} G \arrow[d, hook] & \mathbb{C} \rtimes_{\mathrm{u}} G \arrow[l] \arrow[r] \arrow[d] & \mathbb{C} \rtimes_{\mu} G \arrow[d] \\
A \rtimes_{\alpha,\mathrm{r}} G \arrow[r, equal] & A \rtimes_{\alpha,\mathrm{u}} G \arrow[r, equal] & A \rtimes_{\alpha,\mu} G
\end{tikzcd}$$

from which it follows that the image of

$$\mathbb{C} \rtimes_{\mu} G \to A \rtimes_{\alpha,\mu} G$$

is the reduced group C^*-algebra. Hence $\mathbb{C} \rtimes_{\mu} G$ dominates $C^*_{\mathrm{r}}(G)$. Such an (A, α) exists whenever G is discrete and exact: indeed, one may take $A = C(M)$ where M is a minimal subsystem of the Stone-Čech compactification of G, as discussed in [27, Sections 1.4–1.5], and apply the result of [1] combined with the fact that the action of G on M is amenable, and thus $C(M) \rtimes_{\mathrm{u}} G = C(M) \rtimes_{\mathrm{r}} G$ [5, Theorem 4.3.4]. Plausibly one could adapt such an argument to exact locally compact groups, but the necessary ingredients seem to be missing from the literature.[2] It is not known whether an A with the above properties can exist for non exact groups G.

5 The Minimal Exact Correspondence Functor

One of the main motivations for considering exotic crossed products is to better understand counterexamples to the famous Baum-Connes conjecture. The conjecture, with coefficients, in its original form claimed that a certain *assembly map*

$$\mathrm{as}^{\mathrm{r}}_{(G,A)} : K^{\mathrm{top}}_*(G; A) \to K_*(A \rtimes_{\mathrm{r}} G)$$

should always be an isomorphism. We refer to [2] for the definition of the assembly map and the group $K^{\mathrm{top}}_*(G; A)$. We will not try to summarise the conjecture here, but note that all known failures of the Baum-Connes conjecture for groups with coefficients [20] are essentially all down to failures of exactness as in the following definition.

Definition 5.1 A crossed product functor μ is exact if for any short exact sequence of G-C^*-algebras

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0\ ,$$

[2] Added in proof: these ingredients recently appeared for second countable G in a paper of Brodzki, Cave, and Li, arXiv.1603.01829.

the corresponding sequence of crossed products

$$0 \longrightarrow I \rtimes_\mu G \longrightarrow A \rtimes_\mu G \longrightarrow B \rtimes_\mu G \longrightarrow 0 \tag{5.1}$$

is still exact.

The universal crossed product is always exact, but infamous examples due to Gromov [17] (and recently given quite a satisfactory treatment by Osajda and others [33]) show that the reduced crossed product can fail to be exact for some groups. Groups for which the reduced crossed product is exact were first studied by Kirchberg and Wassermann [28], who called them *exact* groups. The class of exact groups includes many interesting classes of groups, such as almost connected groups, discrete linear groups, amenable groups, and hyperbolic groups.

In the sequence in line (5.1) above, the map to $B \rtimes_\mu G$ is always surjective (as its image contains the dense *-subalgebra $C_c(G,B)$), and the composition of the two central *-homomorphisms is zero (by functoriality). Hence the potential failures of exactness are that the map $I \rtimes_\mu G \to A \rtimes_\mu G$ might not be injective, and that the kernel of the map $A \rtimes_\mu G \to B \rtimes_\mu G$ might properly contain the image of the map $I \rtimes_\mu G \to A \rtimes_\mu G$. The first of these is just the ideal property that we already considered in the previous section. The second however, is independent of any of the (other) properties we have considered so far: indeed, BG crossed products are always exact, while generally failing most of the other properties in the previous section. On the other hand, KLQ crossed products have all of the properties considered in the previous section, but can fail to be exact as the reduced crossed product can fail to be exact.

Baum, Guentner, and Willett proposed to 'fix' the Baum-Connes conjecture (for groups, with coefficients) by replacing the reduced crossed product that is traditionally used to define the conjecture with an exotic crossed product that is automatically exact. Indeed, there is a canonical way to construct an assembly

$$\mathrm{as}^{\mu}_{(G,A)} : K_*^{\mathrm{top}}(G;A) \to K_*(A \rtimes_\mu G) \tag{5.2}$$

for any crossed-product functor $\rtimes_\mu$, since the original map always factors over the K-theory $K_*(A \rtimes_{\mathrm{u}} G)$ of the universal crossed product. It is well known that the assembly map for the maximal crossed product, which is exact, fails to be an isomorphism in general. So the exact crossed-product functor for the reformulated conjecture should be as close to the reduced one as possible. For compatibility with Morita equivalences (and also to ensure the existence of a descent functor in E-theory) they require their functor to in addition have the following property.

Definition 5.2 Let $\mathcal{K}_G$ denote the compact operators on $L^2(G) \otimes l^2(\mathbb{N})$, equipped with the adjoint action Λ coming from the tensor product of the regular representation and the trivial representation on $l^2(\mathbb{N})$. As for any G-C^*-algebra (A,α), the

G-C^*-algebras $(A \otimes \mathscr{K}_G, \alpha \otimes \Lambda)$ and $(A \otimes \mathscr{K}_G, \alpha \otimes \mathrm{id})$ are equivariantly Morita equivalent, strong Morita compatibility of the universal crossed product can be used to give a canonical isomorphism

$$(A \otimes \mathscr{K}_G) \rtimes_{\alpha \otimes \Lambda, \mathrm{u}} G \cong (A \rtimes_{\alpha, \mathrm{u}} G) \otimes \mathscr{K}_G$$

(see [9, Corollary 5.4]). A crossed product μ is *Morita compatible* if this descends to an isomorphism

$$(A \otimes \mathscr{K}_G) \rtimes_{\alpha \otimes \Lambda, \mu} G \cong (A \rtimes_{\alpha, \mu} G) \otimes \mathscr{K}_G.$$

In order to define the exotic crossed products used in the reformulations of the Baum-Connes conjecture, note that there is a natural order on the collection of all crossed products defined by saying $\rtimes_\mu \geq \rtimes_\nu$ if the identity on $C_c(G, A)$ extends to a *-homomorphism $A \rtimes_\mu G \to A \rtimes_\nu G$ for all G-C^*-algebras A. Using an idea of Kirchberg, Baum, Guentner and Willett prove the following theorem.

Theorem 5.3 *There is a minimal exact and Morita compatible crossed product functor* $\rtimes_{\mathscr{E}}$ *with respect to the order above.*

This crossed product can be used to reformulate the Baum-Connes conjecture with coefficients, asking the assembly map $\mathrm{as}^{\mathscr{E}}_{(G,A)}$ *of (5.2) to be an isomorphism for all G-algebras* (A, α)*, in such a way that the reformulated conjecture has no (at time of writing!) known counterexamples, and such that some of the counterexamples to the old conjecture are confirming examples for the reformulated conjecture.*

The minimal crossed product $\rtimes_{\mathscr{E}}$ is natural to consider here as it is in some sense closest to the reduced crossed product, and as it does not change the conjecture for exact groups. It also has the advantage that it does not suffer from the property (T) obstructions to the version of the Baum-Connes conjecture defined using the universal crossed product [3, Corollary 5.7].

Using the results of Sect. 4, one can prove an analogue of this result in the setting of correspondence functors [9, Section 8].

Theorem 5.4 *There is a minimal exact correspondence functor* $\rtimes_{\mathscr{E}_{\mathfrak{Corr}}}$ *with respect to the order above.*

Moreover, this functor agrees with the minimal exact Morita compatible functor on the category of separable G-C^**-algebras and equivariant* **-homomorphisms.*

Combined with the results of the previous section, this shows that one can also use the full power of KK^G theory to study the reformulated Baum-Connes conjecture.

6 Restriction, Extension, and Induction to and from Subgroups

Suppose that H is a closed subgroup of G. In this section we want to study relations between crossed-product functors on G and crossed-product functors on H. We shall define in particular a restriction and extension process between functors for G and functors for H.

We start with the restriction process: Suppose that $\rtimes_\mu$ is a crossed-product functor for G and suppose that (A, α) is an H-algebra. Consider the induced G-algebra $(\mathrm{Ind}_H^G(A, \alpha), \mathrm{Ind}\,\alpha)$ in which

$$\mathrm{Ind}_H^G(A, \alpha) := \left\{ F \in C_b(G, A) : \begin{array}{c} \alpha_h(F(sh)) = F(s)\ \forall s \in G, h \in H, \\ \text{and } (sH \mapsto \|F(s)\|) \in C_0(G/H) \end{array} \right\}.$$

The G-action on $\mathrm{Ind}_H^G(A, \alpha)$ is given by $\big(\mathrm{Ind}\,\alpha_s(F)\big)(t) = F(s^{-1}t)$. Now recall Green's imprimitivity theorem (see [14, Theorem B2] or [35]), which provides a natural equivalence bimodule $X(A, \alpha)$ between $\mathrm{Ind}_H^G(A, \alpha) \rtimes_{\mathrm{Ind}\,\alpha,\mathrm{u}} G$ and $A \rtimes_{\alpha,\mathrm{u}} H$. Let

$$I_{\mathrm{Ind}\,\alpha,\mu} := \ker\Big(\mathrm{Ind}_H^G(A, \alpha) \rtimes_{\mathrm{Ind}\,\alpha,\mathrm{u}} G \to \mathrm{Ind}_H^G(A, \alpha) \rtimes_{\mathrm{Ind}\,\alpha,\mu} G \Big).$$

By the Rieffel correspondence between ideals in $\mathrm{Ind}_H^G(A, \alpha) \rtimes_{\mathrm{Ind}\,\alpha,\mathrm{u}} G$ and ideals in $A \rtimes_{\alpha,\mathrm{u}} H$ there is a unique ideal $I_{\alpha,\mu|_H} \subseteq A \rtimes_{\alpha,\mathrm{u}} H$ such that $X(A, \alpha)$ factors through an equivalence bimodule $X_\mu(A, \alpha)$ between $\mathrm{Ind}_H^G(A, \alpha) \rtimes_{\mathrm{Ind}\,\alpha,\mu} G$ and the quotient

$$A \rtimes_{\alpha,\mu|_H} H := (A \rtimes_{\alpha,\mathrm{u}} H)/I_{\alpha,\mu|_H}. \tag{6.1}$$

Definition 6.1 Let $\rtimes_\mu$ be a crossed-product functor for G. Then the assignment $(A, \alpha) \mapsto A \rtimes_{\alpha,\mu|_H} H$, with $A \rtimes_{\alpha,\mu|_H} H$ constructed as above, is called the *restriction* of $\rtimes_\mu$ to H.

Remark 6.2 By the definition of the restricted crossed-product functor the following version of Green's imprimitivity theorem holds automatically: If $\rtimes_\mu$ is a G-crossed-product functor and if $\rtimes_{\mu|_H}$ denotes its restriction to the closed subgroup H of G, then Green's bimodule $X(A, \alpha)$ factors through an equivalence bimodule from $\mathrm{Ind}_H^G(A, \alpha) \rtimes_{\mathrm{Ind}\,\alpha,\mu} G$ to $A \rtimes_{\alpha,\mu|_H} H$. Moreover, this determines the restriction $\rtimes_{\mu|_H}$ and since Green's imprimitivity theorem holds for both full and reduced norms (see [14, Appendix B] for a detailed discussion and references), it follows that $\rtimes_{\mathrm{u}|_H} = \rtimes_{\mathrm{u}}$ and $\rtimes_{\mathrm{r}|_H} = \rtimes_{\mathrm{r}}$. More generally, it follows from [7, Theorem 5.12] that if $\mu = \mu_E$ is a KLQ-crossed-product functor for G corresponding to a G-invariant ideal $E \subseteq B(G)$, then the restriction $\mu|_H$ is the KLQ-functor for H which corresponds to the H-invariant ideal E_H of $B(H)$ which is generated by $E|_H = \{f|_H : f \in E\}$.

Theorem 6.3 *Let $\rtimes_\mu$ be a crossed-product functor for G and let $\rtimes_{\mu|_H}$ be its restriction to H. Then the following are true:*

1. *$(A,\alpha) \mapsto A \rtimes_{\alpha,\mu|_H} H$ is a crossed-product functor for H.*
2. *If $\rtimes_\mu$ has the ideal property, the same holds for $\rtimes_{\mu|_H}$.*
3. *If $\rtimes_\mu$ is a correspondence functor, the same holds for $\rtimes_{\mu|_H}$.*
4. *If $\rtimes_\mu$ is exact, the same holds for $\rtimes_{\mu|_H}$.*

Remark 6.4 Before we give the proof of the theorem, we need to say some words about the connection of the composition $\psi \circ \phi$ of two *-homomorphisms $\phi : A \to B$ and $\psi : B \to C$ and composition in the correspondence category $\mathfrak{Corr}$.

For this observe that ϕ and ψ can be represented by the (nondegenerate) $A-B$ and $B-C$ correspondences $(\phi(A)B, \phi)$ and $(\psi(B)C, \psi)$, respectively. The assignment sending *-homomorphisms to the equivalence classes of these correspondences is functorial, so that the composition $\psi \circ \phi$ is represented by the correspondence $(\phi(A)B \otimes_B \psi(B)C, \phi \otimes 1)$.

Proof Suppose that $\phi : A \to B$ is a *-homomorphism. In order to show that it induces a *-homomorphism $\phi \rtimes_{\mu|_H} H : A \rtimes_{\alpha,\mu|_H} H \to B \rtimes_{\beta,\mu|_H} H$ we need to show that the composition of the quotient map $q_{B,\mu|_H} : B \rtimes_{\beta,\mathrm{u}} H \to B \rtimes_{\beta,\mu|_H} H$ with the descent $\phi \rtimes_{\mathrm{u}} H : A \rtimes_{\alpha,\mathrm{u}} H \to B \rtimes_{\beta,\mathrm{u}} H$ factors through $A \rtimes_{\alpha,\mu|_H} H$.

For this we consider the following diagram in the correspondence category $\mathfrak{Corr}$:

$$\begin{tikzcd}
A \rtimes_{\alpha,\mathrm{u}} H \arrow[r, "X(A,\alpha)^*"] \arrow[d, "\phi\rtimes_{\mathrm{u}}H"'] & \mathrm{Ind}_H^G(A,\alpha) \rtimes_{\mathrm{Ind}\,\alpha,\mathrm{u}} G \arrow[d, "\mathrm{Ind}_H^G\,\phi\rtimes_{\mathrm{u}}G"] \\
B \rtimes_{\beta,\mathrm{u}} H \arrow[r, "X(B,\beta)^*"] \arrow[d, "q_{\mu|_H}"'] & \mathrm{Ind}_H^G(B,\beta) \rtimes_{\mathrm{Ind}\,\beta,\mathrm{u}} G \arrow[d, "q_\mu"] \\
B \rtimes_{\beta,\mu|_H} H \arrow[r, "X_\mu(B,\beta)^*"] & \mathrm{Ind}_H^G(B,\beta) \rtimes_{\mathrm{Ind}\,\beta,\mu} G.
\end{tikzcd}$$

It is shown in [14, Chapter 4] that the upper square of this diagram commutes (in [14] only reduced crossed products are considered, but the same arguments work for full crossed products as well). It follows from the definition of $\rtimes_{\mu|_H}$ that the lower square commutes. Hence the outer square commutes as well. This implies that the kernel $J_{\mathrm{Ind}\,\phi\rtimes_\mu G}$ of the composition of the right vertical arrows in the diagram corresponds to the kernel $J_{\phi\rtimes_{\mu|_H}H}$ of the composition of the left vertical arrows via the Rieffel correspondence induced from $X(A,\alpha)$. By assumption we have $J_{\mathrm{Ind}\,\phi\rtimes_\mu G} \subseteq I_{\mathrm{Ind}\,\alpha,\mu}$. Since the Rieffel correspondence preserves inclusions, it follows that $J_{\phi\rtimes_{\mu|_H}H} \subseteq I_{\alpha,\mu|_H}$, hence the left vertical arrows factor through $A \rtimes_{\alpha,\mu|_H} H$. This proves (1).

Assume now that I is an H-invariant ideal of A with inclusion map $\iota : I \to A$ and let $Q : A \to A/I$ denote the quotient map. It is well known (and trivial to check) that

the corresponding sequence

$$0 \longrightarrow \operatorname{Ind}_H^G(I,\alpha) \xrightarrow{\operatorname{Ind}\iota} \operatorname{Ind}_H^G(A,\alpha) \xrightarrow{\operatorname{Ind}Q} \operatorname{Ind}_H^G(A/I,\alpha) \longrightarrow 0$$

is a short exact sequence of G-algebras. We then get the following commutative diagram in $\mathfrak{Corr}$:

$$\begin{array}{ccc}
I \rtimes_{\alpha,\mu|_H} H & \xrightarrow{X_\mu(I,\alpha)^*} & \operatorname{Ind}_H^G(I,\alpha) \rtimes_{\operatorname{Ind}\alpha,\mu} G \\
{\scriptstyle \iota\rtimes_{\mu|_H}H}\downarrow & & \downarrow{\scriptstyle \operatorname{Ind}_H^G \iota\rtimes_\mu G} \\
A \rtimes_{\alpha,\mu|_H} H & \xrightarrow{X_\mu(A,\beta)^*} & \operatorname{Ind}_H^G(A,\alpha) \rtimes_{\operatorname{Ind}\alpha,\mu} G \\
{\scriptstyle Q\rtimes_{\mu|_H}H}\downarrow & & \downarrow{\scriptstyle \operatorname{Ind}_H^G Q\rtimes_\mu G} \\
(A/I) \rtimes_{\alpha,\mu|_H} H & \xrightarrow{X_\mu(A/I,\beta)^*} & \operatorname{Ind}_H^G(A/I,\alpha) \rtimes_{\operatorname{Ind}\alpha,\mu} G.
\end{array}$$

If $\rtimes_\mu$ has the ideal property, $\operatorname{Ind}_H^G \iota \rtimes_\mu G$ is injective. It follows from the commutativity of the upper square that $\iota \rtimes_{\mu|_H} H$ is injective as well. Hence $\rtimes_{\mu|_H}$ also satisfies the ideal property. If $\rtimes_\mu$ is exact, then the commutativity of the whole diagram implies that $\rtimes_{\mu|_H}$ is exact. This proves (2) and (4).

Finally, the commutativity of the diagram

$$\begin{array}{ccc}
B \rtimes_{\alpha,\mu|_H} H & \xrightarrow{X_\mu(I,\alpha)^*} & \operatorname{Ind}_H^G(B,\alpha) \rtimes_{\operatorname{Ind}\alpha,\mu} G \\
{\scriptstyle \iota\rtimes_{\mu|_H}H}\downarrow & & \downarrow{\scriptstyle \operatorname{Ind}_H^G \iota\rtimes_\mu G} \\
A \rtimes_{\alpha,\mu|_H} H & \xrightarrow{X_\mu(A,\beta)^*} & \operatorname{Ind}_H^G(A,\alpha) \rtimes_{\operatorname{Ind}\alpha,\mu} G
\end{array}$$

where B is an H-invariant hereditary subalgebra of A—which makes $\operatorname{Ind}_H^G(B,\alpha)$ a G-invariant hereditary subalgebra of $\operatorname{Ind}_H^G(A,\alpha)$—and where $\iota : B \to A$ denotes the inclusion map, implies that the hereditary subalgebra property passes from $\rtimes_\mu$ to $\rtimes_{\mu|_H}$. By Theorem 4.9 in [9] this implies (3) and finishes the proof. □

As a direct consequence of the above result, we get the following well-known, but non-trivial result (the original proof by Kirchberg and Wassermann in [29] uses similar ideas as used in the above theorem):

Corollary 6.5 *Every closed subgroup of an exact group is also exact.*

Proof A locally compact group G is exact if and only if the minimal exact correspondence functor $\rtimes_{\mathcal{E}^G_{\mathfrak{Corr}}}$ equals the reduced G-crossed-product functor $\rtimes_{\mathrm{r}}^G$. But this implies that if $H \subseteq G$ is a closed subgroup, then $\rtimes_{\mathrm{r}}^G|_H = \rtimes_{\mathrm{r}}^H$ is exact by Theorem 6.3(4). □

The restriction of a crossed-product functor to a subgroup also has good properties with respect to the Baum-Connes assembly map:

Proposition 6.6 *Suppose that $\rtimes_\mu$ is a crossed-product functor for a second countable locally compact group G and let H be a closed subgroup of G. Let (A,α) be any separable H-algebra. Then the following are equivalent:*

1. *The assembly map $\mathrm{as}^{\mu|_H}_{(A,H)}: K_*^{\mathrm{top}}(H,A) \to K_*(A \rtimes_{\alpha,\mu|_H} H)$ is an isomorphism.*
2. *the assembly map $\mathrm{as}^{\mu}_{(\mathrm{Ind}_H^G A,G)}: K_*^{\mathrm{top}}(G,\mathrm{Ind}_H^G A) \to K_*(\mathrm{Ind}_H^G A \rtimes_{\mathrm{Ind}\,\alpha,\mu} G)$ is an isomorphism.*

In particular, if G satisfies BC for all $\rtimes_\mu$-crossed products, then H satisfies BC for all $\rtimes_{\mu|_H}$-crossed products.

Proof It is shown in [10, Theorem 2.2] that there is an isomorphism $\mathrm{Ind}_H^G: K_*^{\mathrm{top}}(H,A) \to K_*^{\mathrm{top}}(G;\mathrm{Ind}_H^G A)$ which by [10, Proposition 2.3] commutes with the assembly maps for H and G in the sense that the following diagram commutes:

$$\begin{tikzcd}
K_*^{\mathrm{top}}(H,A) \arrow[r, "\mathrm{as}^{\mathrm{r}}"] \arrow[d, "\mathrm{Ind}_H^G"'] & K_*(A \rtimes_{\alpha,\mathrm{r}} H) \arrow[d, "\sim_M"] \\
K_*^{\mathrm{top}}(G,\mathrm{Ind}_H^G A) \arrow[r, "\mathrm{as}^{\mathrm{r}}"'] & K_*(\mathrm{Ind}_H^G A \rtimes_{\mathrm{Ind}\,\alpha,\mathrm{r}} G),
\end{tikzcd}$$

where the right vertical arrow is induced from Green's Morita equivalence. But the arguments used in the proof of [10, Proposition 2.3] can easily be adapted to show that the following diagram

$$\begin{tikzcd}
K_*^{\mathrm{top}}(H,A) \arrow[r, "\mathrm{as}^{\mu|_H}"] \arrow[d, "\mathrm{Ind}_H^G"'] & K_*(A \rtimes_{\alpha,\mu|_H} H) \arrow[d, "\sim_M"] \\
K_*^{\mathrm{top}}(G,\mathrm{Ind}_H^G A) \arrow[r, "\mathrm{as}^{\mu}"'] & K_*(\mathrm{Ind}_H^G A \rtimes_{\mathrm{Ind}\,\alpha,\mu} G).
\end{tikzcd}$$

commutes as well. This finishes the proof. □

In view of the above proposition it is interesting to study the question whether the restriction of the minimal exact correspondence functor for G to a closed subgroup H will always give the minimal exact correspondence functor for H, since this would then imply that the reformulated conjecture passes to closed subgroups. We shall show in the following section that this is indeed true whenever H is normal in G, but we do not know the answer in general.

We are now going to construct crossed-product functors for G out of crossed-product functors for a closed subgroup H. There are actually (at least) two possibilities for doing this. We start with what we call the *extension* of a crossed-product functor to G:

Definition 6.7 Suppose H is a closed subgroup of G and let $(B, \beta) \mapsto B \rtimes_{\beta,\nu} H$ be a crossed-product functor for H. Then, if (A, α) is a G-algebra, we define the crossed product $A \rtimes_{\alpha,\mathrm{ext}\nu} G$ as $A \rtimes_{\alpha,\mathrm{ext}\nu} G := (A \rtimes_{\alpha,\mathrm{u}} G)/J_\nu$ with

$$J_\nu = \cap\{\ker(\pi \rtimes u) : (\pi, u) \in \mathrm{Rep}(A \rtimes_{\alpha,\mathrm{u}} G) \text{ such that } \pi \rtimes u|_H \in \mathrm{Rep}(A \rtimes_{\alpha,\nu} H)\}.$$

We call $\rtimes_{\mathrm{ext}\nu}$ the *extension* of $\rtimes_\nu$ to G.

In other words, $A \rtimes_{\mathrm{ext}\nu} G$ is the "largest" G-crossed product such that all representations of $A \rtimes_{\mathrm{ext}\nu} G$ restrict to representations of $A \rtimes_\nu H$. To get a feeling for it observe that the extension of the universal crossed-product functor on H is the universal crossed-product functor for G, but the extension of the reduced crossed-product functor on H will rarely be the reduced crossed-product functor for G. In fact, if H is amenable, it will always be the universal one.

Theorem 6.8 *Let $\rtimes_\nu$ be a crossed-product functor for H and let $\rtimes_{\mathrm{ext}\nu}$ be the extension of $\rtimes_\nu$ to G. Then the following are true:*

1. *$(A, \alpha) \mapsto A \rtimes_{\alpha,\mathrm{ext}\nu} H$ is a crossed-product functor for G.*
2. *If $\rtimes_\nu$ has the ideal property, the same holds for $\rtimes_{\mathrm{ext}\nu}$.*
3. *If $\rtimes_\nu$ is a correspondence functor, the same holds for $\rtimes_{\mathrm{ext}\nu}$.*
4. *If $\rtimes_\nu$ is exact, the same holds for $\rtimes_{\mathrm{ext}\nu}$.*

Proof In all four cases, we just show that the property of interest can be reformulated in terms of covariant pairs; having done this, it is then straightforward to verify that if ν has the given property, then $\mathrm{ext}\nu$ also does.

For functoriality, let $\phi : A \to B$ be an L-equivariant *-homomorphism for some group L. Then for given quotients $A \rtimes_\mu L$ and $B \rtimes_\mu L$ of the universal crossed products, $\phi \rtimes_{\mathrm{u}} L : A \rtimes_{\mathrm{u}} L \to B \rtimes_{\mathrm{u}} L$ descends to a *-homomorphism $A \rtimes_\mu L \to B \rtimes_\mu L$ if and only if for every covariant pair (π, u) for (B, L) that extends to $B \rtimes_\mu L$, the covariant pair $(\pi \circ \phi, u)$ extends to $A \rtimes_\mu L$.

For the ideal property, let I be an L-invariant ideal in some A and $\rtimes_\mu$ a crossed product functor for L. For a nondegenerate representation π of I, let $\tilde{\pi}$ denote the canonical extension to A. Note that $\rtimes_\mu$ has the ideal property if and only if $\tilde{\pi} \rtimes u$ extends to $A \rtimes_\mu L$ whenever the covariant pair (π, u) integrates to a representation of $A \rtimes_\mu L$.

To check the correspondence functor property, we work with the projection property. Let A be an L-algebra and p an L-invariant projection in the multiplier algebra of A. Let $\rtimes_\mu$ be a crossed product. For a nondegenerate representation π of A on a Hilbert space $\mathscr{H}$, let $\pi|_p$ denote the restriction of π to the corner pAp acting on $\pi(p)\mathscr{H}$ (where we have also used π for the canonical extension of π to the multiplier algebra of A). Recall from [9, Corollary 8.6] that the crossed product $\rtimes_\mu$ has the projection property if and only if for any such A and p and any covariant pair (π, u) that integrates to $A \rtimes_\mu L$, the covariant pair $(\pi|_p, u)$ for (pAp, L) integrates to $pAp \rtimes_\mu L$.

Finally, let

$$0 \to I \to A \to B \to 0$$

be a short exact sequence of L-C^*-algebras and $\rtimes_\mu$ a crossed product. As we have already considered the ideal property, it remains to characterise exactness of the sequence

$$0 \to I \rtimes_\mu L \to A \rtimes_\mu L \to B \rtimes_\mu L \to 0$$

at the middle term. For a representation π of A that contains I in its kernel, let $\dot{\pi}$ denote the representation of B canonically induced by π. Note that the sequence above is exact at the middle term precisely when for any representation (π, u) that integrates to $A \rtimes_\mu L$ such that π contains I in the kernel, the representation $(\dot{\pi}, u)$ integrates to $B \rtimes_\mu L$.

As a sample, we give the proof of (3) and leave the other assertions to the reader. For this let $p \in \mathscr{M}(A)$ be a G-invariant projection and let (π, u) be a covariant representation of (A, G, α) that integrates to $A \rtimes_{\mathrm{ext}\nu} G$. Then, by definition of $\rtimes_{\mathrm{ext}\nu}$, $(\pi, u|_H)$ integrates to $A \rtimes_\nu H$. The projection property for $\rtimes_\nu$ implies that $(\pi|_p, u|_H)$ integrates to $pAp \rtimes_\nu H$, which then implies that $(\pi|_p, u)$ integrates to $pAp \rtimes_{\mathrm{ext}\nu} G$. □

We are now going to describe an alternative procedure to construct a crossed-product functor on G from a functor $\rtimes_\nu$ on a closed subgroup H of G which we call the *induced* crossed-product functor. For this recall that if we start with a G-algebra (A, α) and restrict the action to H, then the induced algebra $\mathrm{Ind}_H^G(A, \alpha)$ is G-isomorphic to $A \otimes C_0(G/H)$ equipped with the diagonal action, where G acts on G/H by the left-translation action τ (e.g., see [12, Remark 6.1]). Then Green's imprimitivity theorem provides us with an equivalence bimodule $X(A, \alpha)$ between $A \rtimes_{\alpha,\mathrm{u}} H$ and $(A \otimes C_0(G/H)) \rtimes_{\alpha\otimes\tau,\mathrm{u}} G$. Given a crossed-product functor $\rtimes_\nu$ for H with corresponding ideal $I_{\alpha,\nu} \subseteq A \rtimes_{\alpha,\mathrm{u}} H$, we can consider the ideal $I_{A\otimes C_0(G/H),\tilde{\nu}}$ in $(A \otimes C_0(G/H)) \rtimes_{\alpha\otimes\tau,\mathrm{u}} G$ which corresponds to $I_{A,\nu}$ via the Rieffel correspondence for the bimodule $X(A, \alpha)$. Let $(A \otimes C_0(G/H)) \rtimes_{\alpha\otimes\tau,\tilde{\nu}} G$ denote the quotient of the full crossed product by this ideal. Let $j_A : A \to \mathscr{M}(A \otimes C_0(G/H))$ denote the canonical inclusion and let

$$q_{\tilde{\nu}} : (A \otimes C_0(G/H)) \rtimes_{\alpha\otimes\tau,\mathrm{u}} G \to (A \otimes C_0(G/H)) \rtimes_{\alpha\otimes\tau,\tilde{\nu}} G$$

denote the quotient map. We then introduce the following:

Definition 6.9 Let H be a closed subgroup of G and let $\rtimes_\nu$ be a crossed-product functor for H. Then the *induced crossed-product functor* $\rtimes_{\mathrm{Ind}\,\nu}$ for G is defined as

$$A \rtimes_{\alpha,\mathrm{Ind}\,\nu} G := q_{\tilde{\nu}} \circ (j_A \rtimes_{\mathrm{u}} G)\big(A \rtimes_{\alpha,\mathrm{u}} G\big) \subseteq \mathscr{M}\big((A \otimes C_0(G/H)) \rtimes_{\alpha\otimes\tau,\tilde{\nu}} G\big).$$

Example 6.10 The crossed-product functor induced from the reduced H-crossed product $\rtimes_{\mathrm{r}}^{H}$ is the reduced G-crossed product $\rtimes_{\mathrm{r}}^{G}$. This is because Green's imprimitivity bimodule $X_H^G(A,\alpha)$ factors through the reduced norms and the canonical homomorphism $A \rtimes_{\mathrm{r}} G \to \mathscr{M}((A \otimes C_0(G/H)) \rtimes_{\mathrm{r}} G)$ is an embedding for every G-algebra A.

In particular, if H is amenable (and in particular if H is the trivial group), we always get the reduced G-crossed product functor by induction from the (unique) H-crossed product functor. This also means that induction from the universal norm does not always give the universal norm.

We have the following general properties for the induced crossed-product functors.

Theorem 6.11 *Let $\rtimes_\nu$ be a crossed-product functor for H. Then the following are true:*

1. *$(A,\alpha) \mapsto A \rtimes_{\alpha,\mathrm{Ind}\,\nu} G$ is a crossed-product functor for G.*
2. *If $\rtimes_\nu$ has the ideal property, the same holds for $\rtimes_{\mathrm{Ind}\,\nu}$.*
3. *If $\rtimes_\nu$ is a correspondence functor, the same holds for $\rtimes_{\mathrm{Ind}\,\nu}$.*
4. *If $\rtimes_\nu$ is exact and G/H is compact, then $\rtimes_{\mathrm{Ind}\,\nu}$ is exact as well.*

Proof Exactly the same arguments as used in the proof of Theorem 6.3 show that the assignment $(A,G,\alpha) \mapsto (A \otimes C_0(G/H)) \rtimes_{\alpha\otimes\tau,\tilde{\nu}} G$ is a functor from the category of G-algebras into the category of C^*-algebras which preserves suitable versions of the ideal property or exactness, if this holds for the given functor on H. Then similar arguments as used in the proof of [9, Corollary 4.20] imply items (1), (2) and (3). For the proof of (4) we consider the diagram

$$\begin{tikzcd}
I \rtimes_\nu H \arrow[r] \arrow[d, "\sim_M"', "\cong"] & A \rtimes_\nu H \arrow[r] \arrow[d, "\sim_M"', "\cong"] & A/I \rtimes_\nu H \arrow[d, "\cong"', "\sim_M"] \\
C(G/H,I) \rtimes_{\tilde{\nu}} G \arrow[r] & C(G/H,A) \rtimes_{\tilde{\nu}} G \arrow[r] & C(G/H,A/I) \rtimes_{\tilde{\nu}} G \\
I \rtimes_{\mathrm{Ind}\,\nu} G \arrow[r] \arrow[u, "\iota\rtimes G"] & A \rtimes_{\mathrm{Ind}\,\nu} G \arrow[r] \arrow[u, "\iota\rtimes G"] & A/I \rtimes_{\mathrm{Ind}\,\nu} G \arrow[u, "\iota\rtimes G"']
\end{tikzcd}$$

in which $\iota : A \to C(G/H,A)$ maps $a \in A$ to $a \cdot 1_{G/H}$. The upper horizontal row is exact by exactness of $\rtimes_\nu$ and then the middle horizontal row is exact since the upper half of the diagram commutes in the correspondence category (compare with the proof of Theorem 6.3). But then a standard argument (see proof of Theorem 8.1 in [12]), using an approximate unit of $C(G/H,I) \rtimes_{\tilde{\nu}} G$ consisting of elements in $C_c(G,I) \subseteq I \rtimes_{\mathrm{Ind}\,\nu} G$ shows that the lower horizontal row is also exact. This finishes the proof. □

Remark 6.12 If $H = \{e\}$ is the trivial subgroup of G, then the (unique) G-crossed-product functor induced from $\{e\}$ is just the reduced crossed-product functor (see Example 6.10). In particular, the induced functor will not be exact if G is not exact, which shows that a functor induced from an exact crossed-product functor does not have to be exact in general.

Lemma 6.13 *Let H be a closed subgroup of G and let $\rtimes_\mu$ be a crossed-product functor on G. Then $\rtimes_{\mathrm{Ind}(\mu|_H)} = \rtimes_{\mu_{C_0(G/H)}}$, where $\rtimes_{\mu_{C_0(G/H)}}$ is the functor constructed from $\rtimes_\mu$ by tensoring with $D = C_0(G/H)$ as in [9, Corollary 4.20].*

Proof Let (A, α) be a G-algebra. Then $\mathrm{Ind}_H^G A$ is isomorphic to $A \otimes C_0(G/H)$ via the isomorphism which sends a function $F \in \mathrm{Ind}_H^G A$ to the function $(g \mapsto \alpha_g(F(g))) \in C_0(G/H, A)$. It follows that $A \rtimes_{\mu|_H} H$ is the unique quotient of $A \rtimes_{\mathrm{u}} H$ such that Green's $(A \otimes C_0(G/H)) \rtimes_{\mathrm{u}} G - A \rtimes_{\mathrm{u}} H$ equivalence bimodule factors through an $(A \otimes C_0(G/H)) \rtimes_\mu G - A \rtimes_{\mu|_H} H$ equivalence bimodule. But it follows then from the definition of the induced functor $\rtimes_{\mathrm{Ind}(\mu|_H)}$ that $A \rtimes_{\mathrm{Ind}(\mu|_H)} G$ is precisely the quotient of $A \rtimes_{\mathrm{u}} G$ by the kernel of the canonical homomorphism from $A \rtimes_{\mathrm{u}} G$ into $\mathcal{M}\big((A \otimes C_0(G/H)) \rtimes_\mu G\big)$, hence it coincides with $A \rtimes_{\mu_{C_0(G/H)}} G$. $\square$

Corollary 6.14 *Let H be a cocompact closed subgroup of G. Then*

$$\rtimes_{\mathrm{Ind}\,\mathscr{E}^H_{\mathfrak{Corr}}} = \rtimes_{\mathscr{E}^G_{\mathfrak{Corr}}}.$$

Proof It follows from [9, Corollary 8.9] together with Lemma 6.13 that the functors $\rtimes_{\mathrm{Ind}(\mathscr{E}^G_{\mathfrak{Corr}}|_H)}$ and $\rtimes_{\mathscr{E}^G_{\mathfrak{Corr}}}$ coincide. Since $\rtimes_{\mathscr{E}^G_{\mathfrak{Corr}}|_H}$ is an exact correspondence functor for H, it dominates $\rtimes_{\mathscr{E}^H_{\mathfrak{Corr}}}$. Hence $\rtimes_{\mathscr{E}^G_{\mathfrak{Corr}}} = \rtimes_{\mathrm{Ind}(\mathscr{E}^G_{\mathfrak{Corr}}|_H)}$ dominates $\rtimes_{\mathrm{Ind}\,\mathscr{E}^H_{\mathfrak{Corr}}}$. Since $\rtimes_{\mathrm{Ind}\,\mathscr{E}^H_{\mathfrak{Corr}}}$ is an exact correspondence functor by Theorem 6.11, it dominates $\rtimes_{\mathscr{E}^G_{\mathfrak{Corr}}}$, hence both must be equal. $\square$

Again, the above result can be used to yield the following well-known consequence:

Corollary 6.15 *If a locally compact group G contains an exact cocompact closed subgroup H, then G is exact.*

Proof If H is exact, then it means that $\rtimes_{\mathscr{E}^H_{\mathfrak{Corr}}} = \rtimes_{\mathrm{r}}^H$ is the reduced crossed-product functor. But since the induced G-crossed-product functor from $\rtimes_{\mathrm{r}}^H$ is $\rtimes_{\mathrm{r}}^G$ (see Example 6.10), it follows from Corollary 6.14 that $\rtimes_{\mathscr{E}^G_{\mathfrak{Corr}}} = \rtimes_{\mathrm{r}}^G$, so G is exact. $\square$

7 Normal Subgroups

In this section we want to show that if N is a closed normal subgroup of G and if $\rtimes_{\mathscr{E}} := \rtimes_{\mathscr{E}^G_{\mathfrak{Corr}}}$ denotes the minimal exact correspondence functor for G, then the restriction $\rtimes_{\mathscr{E}|_N}$ of $\rtimes_{\mathscr{E}}$ to N is the minimal exact correspondence functor for

N. Thus, as a consequence, it follows from Proposition 6.6 that the validity of the reformulated version of the Baum-Connes conjecture due to Baum, Guentner, and Willett will pass to closed normal subgroups. In order to prove the result we need some preparations. As a first step, we show that the minimal crossed-product functor behaves well with respect to automorphisms of the group:

Lemma 7.1 *Let $\varphi : G \to G$ be an automorphism of the locally compact group G and let $\alpha : G \to \mathrm{Aut}(A)$ be an action. Let $\alpha^\varphi : G \to \mathrm{Aut}(A)$ be the action $\alpha^\varphi := \alpha \circ \varphi^{-1}$ and let $\delta_\varphi \in (0,\infty)$ be the module of φ for the Haar measure of G, i.e., we have $\int_G \psi(g)\,dg = \delta_\varphi \int_G \psi(\varphi^{-1}(g))\,dg$ for all $\psi \in C_c(G)$. Then the mapping*

$$\Phi : C_c(G,A,\alpha) \to C_c(G,A,\alpha^\varphi);\ \Phi(f)(g) = \delta_\varphi f(\varphi^{-1}(g))$$

extends to *-isomorphisms

$$\begin{aligned}\Phi_{\mathrm{u}} &: A \rtimes_{\alpha,\mathrm{u}} G \xrightarrow{\sim} A \rtimes_{\alpha^\varphi,\mathrm{u}} G,\\ \Phi_{\mathrm{r}} &: A \rtimes_{\alpha,\mathrm{r}} G \xrightarrow{\sim} A \rtimes_{\alpha^\varphi,\mathrm{r}} G, \quad \text{and}\\ \Phi_{\mathscr{E}} &: A \rtimes_{\alpha,\mathscr{E}} G \xrightarrow{\sim} A \rtimes_{\alpha^\varphi,\mathscr{E}} G.\end{aligned}$$

Proof It is straightforward to check that Φ is a *-isomorphism between the dense *-subalgebras $C_c(G,A,\alpha)$ and $C_c(G,A,\alpha^\varphi)$ of the respective crossed products (the entries α and α^φ indicate the different operations on $C_c(G,A)$). To see that it extends to an isomorphism $\Phi_{\mathrm{u}} : A\rtimes_{\alpha,\mathrm{u}}G \xrightarrow{\sim} A\rtimes_{\alpha^\varphi,\mathrm{u}}G$ we just observe that we have a bijective correspondence between covariant representations of (A,G,α) and of (A,G,α^φ) given by $(\pi,u) \mapsto (\pi,u^\varphi)$ with $u^\varphi := u \circ \varphi^{-1}$ such that $\pi \rtimes u^\varphi(\Phi(f)) = \pi \rtimes u(f)$. This implies that

$$\|\Phi(f)\|_{\mathrm{u}} = \sup_{(\pi,u)} \|\pi \rtimes u^\varphi(\Phi(f))\| = \sup_{(\pi,u)} \|\pi \rtimes u(f)\| = \|f\|_{\mathrm{u}}.$$

For the reduced crossed products recall from Definition 3.1 the construction of the regular representation $\Lambda = \pi \rtimes (\lambda \otimes 1)$ of (A,G,α) on the Hilbert A-module $L^2(G,A)$ with $(\pi(a)\xi)(g) = \alpha_{g^{-1}}(a)\xi(g)$ for $a \in A, g \in G$ and $\xi \in L^2(G,A)$. Similarly, the C*-part π^φ of the regular representation $\Lambda^\varphi = \pi^\varphi \rtimes (\lambda \otimes 1)$ of (A,G,α^φ) is given by $(\pi^\varphi(a)\xi)(g) = \alpha_{\varphi^{-1}(g^{-1})}(a)\xi(g)$. An easy computation then shows that the unitary operator

$$W : L^2(G,A) \to L^2(G,A);\ W\xi(g) = \sqrt{\delta_\varphi}\,\xi(\varphi^{-1}(g))$$

intertwines the regular representation Λ^φ with $\Lambda_A \rtimes \Lambda_G^\varphi$. Hence, up to unitary equivalence, the above described correspondence of covariant representations of

(A, G, α) and (A, G, α^φ) sends Λ to Λ^φ which proves that Φ extends to an isomorphism of the reduced crossed products.

To see that Φ also extends to an isomorphism

$$\Phi_{\mathscr{E}} : A \rtimes_{\alpha,\mathscr{E}} G \xrightarrow{\sim} A \rtimes_{\alpha^\varphi,\mathscr{E}} G$$

we argue as follows: Let $\tilde{\Phi} : A \rtimes_{\alpha,\mathrm{u}} G \to A \rtimes_{\alpha^\varphi,\mathscr{E}} G$ denote the surjective *-homomorphism given by composing Φ_{u} with the quotient map

$$q^\varphi : A \rtimes_{\alpha^\varphi,\mathrm{u}} G \twoheadrightarrow A \rtimes_{\alpha^\varphi,\mathscr{E}} G.$$

Let $A \rtimes_{\alpha,\mathscr{E}^\varphi} G := (A \rtimes_{\alpha,\mathrm{u}} G)/\ker\tilde{\Phi}$. It is straightforward to check that

$$(A, \alpha) \mapsto A \rtimes_{\alpha,\mathscr{E}^\varphi} G$$

is an exact correspondence functor for G, hence it must dominate $A \rtimes_{\alpha,\mathscr{E}} G$ in the sense that the identity on $C_c(G, A)$ induces a surjective *-homomorphism

$$q : A \rtimes_{\alpha,\mathscr{E}^\varphi} G \twoheadrightarrow A \rtimes_{\alpha,\mathscr{E}} G.$$

In other words, this shows that the inverse $\Phi^{-1} : C_c(G, A, \alpha^\varphi) \to C_c(G, A, \alpha)$ extends to a surjective *-homomorphism $A \rtimes_{\alpha^\varphi,\mathscr{E}} G \twoheadrightarrow A \rtimes_{\alpha,\mathscr{E}} G$. Conversely, if we replace φ by its inverse φ^{-1} and apply the above results to the action α^φ, we see that Φ extends to a surjective *-homomorphism $A \rtimes_{\alpha,\mathscr{E}} G \twoheadrightarrow A \rtimes_{\alpha^\varphi,\mathscr{E}} G$. This combines to show that Φ extends to the desired *-isomorphism $\Phi_{\mathscr{E}}$. □

We now want to extend the above result to automorphisms $\varphi := (\varphi_A, \varphi_G)$ of the system (A, G, α). This means that $\varphi_A : A \to A$ is a *-automorphism of A and $\varphi_G : G \to G$ is an automorphism of G such that

$$\alpha_{\varphi_G(s)}(\varphi_A(a)) = \varphi_A(\alpha_s(a)) \quad \forall s \in G, a \in A. \tag{7.1}$$

Note that if $\alpha : G \to \mathrm{Aut}(A)$ is an action and $N \subseteq G$ is a normal subgroup of G, then every $g \in G$ determines an automorphism $\gamma_g := (\alpha_g, C_g)$ of (A, N, α) with $\alpha_g : A \to A$ the given action of the element $g \in G$ and $C_g : N \to N$ the automorphism given by conjugation with g: $C_g(n) = gng^{-1}$. It is well-known that every automorphism φ of (A, G, α) induces automorphisms φ_{u} and φ_{r} on $A \rtimes_{\alpha,\mathrm{u}} G$ and $A \rtimes_{\alpha,\mathrm{r}} G$, respectively, both extending the *-isomorphism $\tilde{\varphi} : C_c(G, A) \to C_c(G, A)$ given by the formula

$$\big(\tilde{\varphi}(f)\big)(s) = \delta_\varphi \varphi_A(f(\varphi_G^{-1}(s))), \quad f \in C_c(G, A), s \in G, \tag{7.2}$$

where δ_φ denotes the module of the automorphism φ_G. The proof follows easily from Lemma 7.1 and the arguments given in the proof of the following lemma in the case of the $\rtimes_{\mathscr{E}}$-crossed products:

Lemma 7.2 *If* $\varphi = (\varphi_A, \varphi_G)$ *is an automorphism of* (A, G, α)*, then the* **-isomorphism* $\tilde{\varphi} : C_c(G, A) \to C_c(G, A)$ *extends to a* **-automorphism* $\varphi_{\mathscr{E}}$ *of* $A \rtimes_{\alpha,\mathscr{E}} G$*.*

Proof Let $\alpha^{\varphi} := \alpha \circ \varphi_G^{-1}$ be the action of G on A 'twisted' by φ_G. Lemma 7.1 shows that $\Phi : C_c(G, A, \alpha) \to C_c(G, A, \alpha^{\varphi})$ given by $\Phi(f)(s) = \delta_{\varphi} f(\varphi_G^{-1}(s))$ extends to an isomorphism $\Phi_{\mathscr{E}} : A \rtimes_{\alpha,\mathscr{E}} G \xrightarrow{\sim} A \rtimes_{\alpha^{\varphi},\mathscr{E}} G$. On the other hand it follows from (7.1) that the automorphism $\varphi_A : A \to A$ is $\alpha^{\varphi} - \alpha$ equivariant. By functoriality of $\rtimes_{\mathscr{E}}$ it therefore induces an isomorphism $\varphi_A \rtimes G : A \rtimes_{\alpha^{\varphi},\mathscr{E}} G \to A \rtimes_{\alpha,\mathscr{E}} G$ given on $C_c(G, A)$ by sending a function f to $\varphi_A \circ f$. The composition $\varphi_{\mathscr{E}} := (\varphi_A \rtimes G) \circ \Phi_{\mathscr{E}}$ is then an automorphism of $A \rtimes_{\alpha,\mathscr{E}} G$ which extends $\tilde{\varphi} : C_c(G, A) \to C_c(G, A)$. ◻

Example 7.3 We should note that the argument of the lemma works for any crossed-product functor $\rtimes_{\mu}$ for G such that an analogue of Lemma 7.1 holds for $\rtimes_{\mu}$, i.e., the isomorphism $\Phi : C_c(G, A, \alpha) \to C_c(G, A, \alpha^{\varphi})$ of that lemma extends to an isomorphism $\Phi_{\mu} : A \rtimes_{\alpha,\mu} G \xrightarrow{\sim} A \rtimes_{\alpha^{\varphi},\mu} G$. Hence it applies in particular for the full and reduced crossed products.

But we should point out that analogues of Lemmas 7.1 and 7.2. do not even hold for all KLQ crossed-product functors. For this let G be any non-amenable group and let $E \subseteq B(G \times G)$ be the weak* closure of the ideal consisting of all coefficient functions ϕ that are supported in a set of the form $G \times K$, where K is a compact subset of G. Then a unitary representation $v : G \times G \to \mathscr{U}(\mathscr{H})$ integrates to the exotic group algebra $C_E^*(G \times G)$ if and only if its restriction to the second factor is weakly contained in λ_G. Let $\varphi : G \times G \to G \times G$ be the flip automorphism and let $v = u_G \otimes \lambda_G$ where u_G denotes the universal representation of G. Then v factors through $C_E^*(G \times G)$ but $v \circ \varphi = \lambda_G \otimes u_G$ does not. Hence φ does not 'extend' to an automorphism of $C_E^*(G \times G)$ and the corresponding KLQ-functor fails Lemmas 7.1 and 7.2.

Lemma 7.4 *Let* $\rtimes_{\mu}$ *be a crossed-product functor for* G *and let* N *be a closed normal subgroup of* G*. Further let* (A, G, α) *be a* G*-system. Then the action* γ^{μ} *of* G *on* $C_c(N, A)$ *defined on the level of* $C_c(N, A)$ *by the formula*

$$(\gamma_g^{\mu}(f))(n) = \delta_g \alpha_g(f(g^{-1}ng)),$$

where δ_g *denotes the module for the automorphism* $C_g(n) = gng^{-1}$ *of* N*, extends to an action* $\gamma^{\mu} : G \to \mathrm{Aut}(A \rtimes_{\alpha,\mu|_N} N)$*.*

Proof Recall that $A \rtimes_{\alpha,\mu|_N} N$ is defined as the quotient $(A \rtimes_{\alpha,\mathrm{u}} N)/J_{\mu}$, where J_{μ} is the ideal in $A \rtimes_{\alpha,\mathrm{u}} N$ which corresponds to the ideal

$$K_{\mu} := \ker(\mathrm{Ind}_N^G A \rtimes_{\mathrm{Ind}\,\alpha,\mathrm{u}} G \to \mathrm{Ind}_N^G A \rtimes_{\mathrm{Ind}\,\alpha,\mu} G)$$

via Green's imprimitivity bimodule. Since A is a G-algebra, the induced algebra $\mathrm{Ind}_N^G A$ is G-isomorphic to $A \otimes C_0(G/N)$ equipped with the diagonal action $\alpha \otimes \tau$, where τ denotes the left-translation action. Let β_g denote the automorphism of $(A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\tau,\mathrm{u}} G$ which is induced from the action $\mathrm{id}_A \otimes \sigma_g$ of $g \in G$ on

$A \otimes C_0(G/N)$, where σ_g denotes the right translation action on $C_0(G/N)$. Note that β_g exists since $\mathrm{id}_A \otimes \sigma_g$ commutes with $\alpha \otimes \tau$. It follows then from functoriality of $\rtimes_\mu$ that β_g factors through an automorphism of $(A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\tau,\mu} G$. In particular, the kernel K_μ of the quotient map from $(A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\tau,\mathrm{u}} G$ onto $(A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\tau,\mu} G$ is β_g-invariant. Now it is part of [13, Theorem 1] that there is a $\beta_g - \gamma_g^{\mathrm{u}}$-compatible automorphism ν_g of $X(A,\alpha)$. Since J_μ corresponds to K_μ via the Rieffel correspondence for the bimodule $X(A,\alpha)$, it follows that $\beta_g(K_\mu)$ corresponds to $\gamma_g^{\mathrm{u}}(J_\mu)$ by Rieffel correspondence. Since this correspondence is one-to-one, and since $K_\mu = \beta_g(K_\mu)$ we also get $J_\mu = \gamma_g^\mu(J_\mu)$. Thus γ_g^{u} factors through an automorphism γ_g^μ of $A \rtimes_{\alpha,\mu|_N} N$. □

We now turn to the problem of showing that the restriction $\rtimes_{\mathscr{E}|_N}$ of the minimal exact correspondence functor $\rtimes_{\mathscr{E}} := \rtimes_{\mathscr{E}^G_{\mathfrak{Corr}}}$ to any closed normal subgroup N of G coincides with the minimal exact correspondence functor $\rtimes_{\mathscr{E}^N_{\mathfrak{Corr}}}$ for N. We need

Lemma 7.5 *Let H be a closed subgroup of G and $\rtimes_\nu$ a crossed-product functor on H. Then, for any G-algebra (A,α), the canonical mapping $i_A \rtimes i_G|_H : A \rtimes_{\mathrm{u}} H \to \mathscr{M}(A \rtimes_{\mathrm{u}} G)$ factors to a well-defined *-homomorphism*

$$i_A^{\mathrm{ext}\nu} \rtimes i_G^{\mathrm{ext}\nu}|_H : A \rtimes_\nu H \to \mathscr{M}(A \rtimes_{\mathrm{ext}\nu} G).$$

If H is open in G, this map takes its image in $A \rtimes_{\mathrm{ext}\nu} G$.

Proof It follows from the definition of $\rtimes_{\mathrm{ext}\nu}$ that $i_A^{\mathrm{ext}\nu} \rtimes i_G^{\mathrm{ext}\nu}|_H$ is well defined and it is clear that it takes image in $A \rtimes_{\mathrm{ext}\nu} G$ if H is open in G. □

Just for the records we show as a corollary of this lemma and Theorem 6.8 that the minimal exact correspondence functor $\rtimes_{\mathscr{E}_{\mathfrak{Corr}}}$ enjoys the following property for arbitrary closed subgroups which is well-known for the maximal and reduced crossed-product functors:

Corollary 7.6 *Let H be a closed subgroup of G, and let (A,α) be a G-C^*-algebra. Then the canonical mapping $i_A \rtimes i_G|_H : A \rtimes_{\mathrm{u}} H \to \mathscr{M}(A \rtimes_{\mathrm{u}} G)$ factors to a well-defined *-homomorphism*

$$i_A^{\mathscr{E}_{\mathfrak{Corr}}} \rtimes i_G^{\mathscr{E}_{\mathfrak{Corr}}}|_H : A \rtimes_{\mathscr{E}^H_{\mathfrak{Corr}}} H \to \mathscr{M}(A \rtimes_{\mathscr{E}^G_{\mathfrak{Corr}}} G).$$

Proof This follows from Lemma 7.5 together with the fact (shown in Theorem 6.8) that the extension $\rtimes_{\mathrm{ext}\mathscr{E}^H_{\mathfrak{Corr}}}$ to G is an exact correspondence functor and hence dominates $\rtimes_{\mathscr{E}^G_{\mathfrak{Corr}}}$. □

Remark 7.7 It is not clear whether the homomorphism

$$i_A^{\mathscr{E}_{\mathfrak{Corr}}} \rtimes i_G^{\mathscr{E}_{\mathfrak{Corr}}}|_H : A \rtimes_{\mathscr{E}^H_{\mathfrak{Corr}}} H \to \mathscr{M}(A \rtimes_{\mathscr{E}^G_{\mathfrak{Corr}}} G)$$

is injective in general, even if H is open in G.

Suppose now that $\rtimes_\mu$ is a crossed-product functor for G and let N be a closed normal subgroup of G. Then, if (A,α) is a G-algebra, we get a canonical *-homomorphism

$$i_A^\mu \rtimes i_G^\mu|_N : A \rtimes_{\alpha,\mathrm{u}} N \to \mathscr{M}((A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\tau,\mu} G), \tag{7.3}$$

given as the composition

$$q_\mu \circ \big((\mathrm{id}_A \otimes 1_{C_0(G/N)}) \rtimes_{\mathrm{u}} G\big) \circ (i_A^{\mathrm{u}} \rtimes i_G^{\mathrm{u}}|_N),$$

with $i_A^{\mathrm{u}} \rtimes i_G^{\mathrm{u}}|_N : A \rtimes_{\alpha,\mathrm{u}} N \to \mathscr{M}(A \rtimes_{\alpha,\mathrm{u}} G)$ the canonical homomorphism, $(\mathrm{id}_A \otimes 1_{C_0(G/N)}) \rtimes_{\mathrm{u}} G : A \rtimes_{\alpha,\mathrm{u}} G \to \mathscr{M}((A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\tau,\mathrm{u}} G)$ induced from $\mathrm{id}_A \otimes 1_{C_0(G/N)} : A \to \mathscr{M}(A \otimes C_0(G/N))$ by functoriality for generalized homomorphisms of $\rtimes_{\mathrm{u}}$, and $q_\mu : (A\otimes C_0(G/N))\rtimes_{\alpha\otimes\tau,\mathrm{u}}G \to (A\otimes C_0(G/N))\rtimes_{\alpha\otimes\tau,\mu}G$ the quotient map.

Lemma 7.8 *The homomorphism of (7.3) factors to a faithful *-homomorphism*

$$i_A^\mu \rtimes i_G^\mu|_N : A \rtimes_{\alpha,\mu|_N} N \to \mathscr{M}((A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\tau,\mu} G).$$

Proof Let $\rho \rtimes v$ be a covariant representation of $A \rtimes_{\alpha,\mathrm{u}} N$ which factors through a faithful representation of $A \rtimes_{\alpha,\mu|_N} N$. Then the induced representation $\mathrm{Ind}_N^G(\rho \rtimes v)$ of (A,G,α) can be defined by first inducing $\rho \rtimes v$ to $(A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\tau,\mathrm{u}} G$ via Green's imprimitivity module $X(A,\alpha)$, which by assumption factors to a faithful representation $\mathrm{Ind}^{X(A,\alpha)}(\rho \rtimes v)$ of $(A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\tau,\mu} G$, and then composing this representation with the canonical embedding

$$(\mathrm{id}_A \otimes 1_{C_0(G/N)}) \rtimes_{\mathrm{u}} G : A \rtimes_{\alpha,\mathrm{u}} G \to \mathscr{M}((A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\tau,\mathrm{u}} G).$$

(e.g., see [12, §6 and §9] and in particular [12, Remark 9.5] for more details). The restriction $\mathrm{Res}_N^G(\mathrm{Ind}_N^G(\rho \rtimes v))$ of $\mathrm{Ind}_N^G(\rho \rtimes v)$ to $A \rtimes_{\alpha,\mathrm{u}} N$ is then given by the composition $\mathrm{Ind}^{X(A,\alpha)}(\rho \rtimes v) \circ (\mathrm{id}_A \otimes 1) \rtimes_{\mathrm{u}} G \circ (i_A^{\mathrm{u}} \rtimes i_G^{\mathrm{u}}|_N)$. Since $\mathrm{Ind}^{X(A,\alpha)}(\rho \rtimes v)$ factors through a faithful representation of $(A\otimes C_0(G/N))\rtimes_{\alpha\otimes\tau,\mu}G$, we may identify it with the quotient map q_μ, and it follows that $I_\mu := \ker\big(\mathrm{Res}_N^G(\mathrm{Ind}_N^G(\rho \rtimes v))\big)$ coincides with the kernel of (7.3). On the other hand, if J_μ denotes the ideal $\ker(\rho \rtimes v) = \ker\big(A \rtimes_{\alpha,\mathrm{u}} N \to A \rtimes_{\alpha,\mu|_N} N\big)$, then by [12, Proposition 9.15] we get

$$\begin{aligned} I_\mu &= \ker(\mathrm{Res}_N^G(\mathrm{Ind}_N^G(\rho \rtimes v))) = \bigcap\{\gamma_g^{\mathrm{u}}(\ker(\rho \rtimes v)) : g \in G\} \\ &= \bigcap\{\gamma_g^{\mathrm{u}}(J_\mu) : g \in G\}, \end{aligned}$$

where $\gamma_g^{\mathrm{u}} : G \to \mathrm{Aut}(A \rtimes_{\alpha,\mathrm{u}} N)$ is the decomposition action of G on $A \rtimes_{\alpha,\mathrm{u}} N$. Since this action factors through $A \rtimes_{\alpha,\mu|_N} N$ by Lemma 7.4, it follows that $\gamma_g^{\mathrm{u}}(J_\mu) = J_\mu$ for all $g \in G$. Thus $I_\mu = J_\mu$. Since I_μ is the kernel of (7.3), the result follows. □

Recall that a $C_0(X)$-algebra is a C^*-algebra A equipped with a nondegenerate *-homomorphism $\varphi : C_0(X) \to \mathscr{Z}\mathscr{M}(A)$, where X is a locally compact space and $\mathscr{Z}\mathscr{M}(A)$ denotes the centre of the multiplier algebra of A. Then, for each $x \in X$, the fibre A_x of A over x is defined as the quotient $A_x := A/I_x$ with $I_x := \varphi(C_0(X\setminus\{x\}))A$. For each $a \in A$ we get an assignment $x \mapsto a_x := a + I_x \in A_x$, hence we may regard A as an algebra of sections of a bundle $\mathscr{A}$ of C^*-algebras over X with fibres A_x. An action $\alpha : G \to \mathrm{Aut}(A)$ is called *fibre-wise*, if $\alpha_g(I_x) = I_x$ for all $g \in G, x \in X$ and hence the action induces actions $\alpha^x : G \to \mathrm{Aut}(A_x)$ for each $x \in X$ in a canonical way. Note that α being fibre-wise is equivalent to α being $C_0(X)$-linear in the sense that $\alpha_g(\varphi(f)a) = \varphi(f)\alpha_g(a)$ for all $g \in G$ and $f \in C_0(X)$. For more information on $C_0(X)$-algebras we refer to [35, Appendix C].

Lemma 7.9 *Suppose that* $\alpha : G \to \mathrm{Aut}(A)$ *is a fibre-wise action of* G *on the* $C_0(X)$*-algebra* A *with structure map* $\varphi : C_0(X) \to \mathscr{Z}\mathscr{M}(A)$*. Suppose further that* $\rtimes_\mu$ *is an exact crossed-product functor for* G*. Then* $A \rtimes_{\alpha,\mu} G$ *is a* $C_0(X)$*-algebra with structure map* $i_A \circ \varphi : C_0(X) \to \mathscr{Z}\mathscr{M}(A \rtimes_{\alpha,\mu} G)$ *and fibres* $A_x \rtimes_{\alpha^x,\mu} G$*.*

Proof As a composition of nondegenerate homomorphisms, $i_A \circ \varphi$ is well defined and nondegenerate, and one easily checks on the generators of $A \rtimes_{\alpha,\mu} G$ that it maps $C_0(X)$ into $\mathscr{Z}\mathscr{M}(A \rtimes_{\alpha,\mu} G)$. To see that $(A \rtimes_{\alpha,\mu} G)_x \cong A_x \rtimes_{\alpha^x,\mu} G$ for all $x \in X$ we first observe that for all open $U \subseteq X$ we get

$$i_A \circ \varphi\big(C_0(U)\big)(A \rtimes_\mu G) = (\varphi(C_0(U))A) \rtimes_\mu G,$$

since by the ideal property (which follows from exactness) both coincide with the closure of $C_c(G, \varphi(C_0(U))A)$ inside $A \rtimes_\mu G$. Hence, if $U = X \setminus \{x\}$ and $I_x := \varphi(C_0(X \setminus \{x\}))A$, it follows from exactness of $\rtimes_\mu$ that

$$A_x \rtimes_{\alpha^x,\mu} G = (A \rtimes_{\alpha,\mu} G)/(I_x \rtimes_{\alpha,\mu} G) = (A \rtimes_{\alpha,\mu} G)_x.$$

□

Remark 7.10 The above lemma is not true for non-exact crossed products in general. Indeed, it is not true for the reduced crossed product of a non-exact group G by [28, Theorem on p. 170].

As a corollary of Lemma 7.9, we get

Lemma 7.11 *Suppose that* $\rtimes_{\mu_i}$*,* $i = 1, 2$*, are two exact crossed-product functors for* G*. Suppose further that* A *is a* $C_0(X)$*-algebra equipped with a fibre-wise action* $\alpha : G \to \mathrm{Aut}(A)$ *such that* $A \rtimes_{\alpha,\mu_1} G = A \rtimes_{\alpha,\mu_2} G$*. Then* $A_x \rtimes_{\alpha^x,\mu_1} G = A_x \rtimes_{\alpha^x,\mu_2} G$ *for all* $x \in X$*.*

Proof This follows from Lemma 7.9 together with the fact that a $C_0(X)$-linear isomorphism between two $C_0(X)$-algebras always induces isomorphisms of the fibres. □

We want to apply the above result to the following example: Suppose that $N \subseteq G$ is a closed normal subgroup of G and let (B, β) be an N-C^*-algebra. Then the induced G-algebra $\mathrm{Ind}_N^G B$ becomes a $C_0(G/N)$-algebra with respect to the nondegenerate *-homomorphism $\varphi : C_0(G/N) \to \mathscr{Z}\mathscr{M}(\mathrm{Ind}_N^G B)$ given by

$$\big(\varphi(f)F\big)(g) = f(gN)F(g), \quad \forall f \in C_0(G/N), F \in \mathrm{Ind}_N^G B, g \in G.$$

Then each fibre $(\mathrm{Ind}_N^G B)_{gN}$ identifies with B via the evaluation map $F \mapsto F(g)$. It is easy to check that the restriction of the G-action $\mathrm{Ind}\,\beta$ to N is $C_0(G/N)$-linear, hence fibre-wise. For the unit fibre eN, it follows that the N-action on the fibre $B \cong (\mathrm{Ind}_N^G B)_{eN}$ coincides with the action $\beta : N \to \mathrm{Aut}(B)$ we started with. This follows from the equation

$$(\mathrm{Ind}\,\beta_n(F))(e) = F(n^{-1}) = \beta_n(F(e)), \quad \forall F \in \mathrm{Ind}_N^G B.$$

Using this, we get

Lemma 7.12 *Suppose that N is a closed normal subgroup of G and that $\rtimes_{\nu_i}$, $i = 1, 2$, are two exact crossed-product functors for N. Suppose further that $\rtimes_{\nu_1} = \rtimes_{\nu_2}$ when restricted to the category of G-C^*-algebras. Then $\rtimes_{\nu_1} = \rtimes_{\nu_2}$.*

Proof Let (B, β) be any N-algebra. Then, by assumption, we have $\mathrm{Ind}_N^G B \rtimes_{\nu_1} N = \mathrm{Ind}_N^G B \rtimes_{\nu_2} N$. But then the above discussion together with Lemma 7.11 implies that $B \rtimes_{\nu_1} N = B \rtimes_{\nu_2} N$. □

We are now ready for the main result of this section:

Theorem 7.13 *Let N be a closed normal subgroup of the locally compact group G. Then the minimal exact correspondence functor $\rtimes_{\mathscr{E}^N_{\mathfrak{Corr}}}$ for N is equal to the restriction $\rtimes_{\mathscr{E}^G_{\mathfrak{Corr}}}|_N$ of the minimal exact correspondence functor $\rtimes_{\mathscr{E}^G_{\mathfrak{Corr}}}$ for G.*

Proof We show that the restriction of $\rtimes_{\mathrm{ext}\mathscr{E}^N_{\mathfrak{Corr}}}$ to N equals $\rtimes_{\mathscr{E}^N_{\mathfrak{Corr}}}$. Since $\rtimes_{\mathrm{ext}\mathscr{E}^N_{\mathfrak{Corr}}}$ is an exact correspondence functor by Theorem 6.8 it dominates $\rtimes_{\mathscr{E}^G_{\mathfrak{Corr}}}$. Hence the desired equation would imply that the restriction of $\rtimes_{\mathscr{E}^G_{\mathfrak{Corr}}}$ to N, which is an exact correspondence functor by Theorem 6.3, would be dominated by $\rtimes_{\mathscr{E}^N_{\mathfrak{Corr}}}$. But then they must coincide.

By Lemma 7.12 it suffices to show that the restriction of $\rtimes_{\mathrm{ext}\mathscr{E}^N_{\mathfrak{Corr}}}$ to N coincides with $\rtimes_{\mathscr{E}^N_{\mathfrak{Corr}}}$ on every G-C^*-algebra (A, α). Let us write $\mu := \mathrm{ext}\mathscr{E}^N_{\mathfrak{Corr}}$. Let $(i_A^\mu, i_{C_0(G/N)}^\mu, i_G^\mu)$ denote the canonical maps from $(A, C_0(G/N), G)$ into the multiplier algebra $\mathscr{M}((A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\tau,\mu} G)$. It follows then from Lemma 7.8 that $i_A^\mu \rtimes i_G^\mu|_N$ factors through a faithful *-homomorphism

$$i_A^\mu \rtimes i_G^\mu|_N : A \rtimes_{\alpha,\mu|_N} N \to \mathscr{M}((A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\tau,\mu} G). \tag{7.4}$$

On the other hand, we know from Lemma 7.5 that we have a homomorphism

$$(i_A^\mu \otimes i_{C_0(G/N)}^\mu) \rtimes i_G^\mu|_N : (A \otimes C_0(G/N)) \rtimes_{\alpha,\mathscr{E}^N_{\mathfrak{Corr}}} N \to \mathscr{M}((A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\tau,\mu} G).$$

Since $\rtimes_{\mu|_N}$ is an exact crossed-product functor, it follows then from minimality of $\rtimes_{\mathcal{E}^N_{\mathfrak{Corr}}}$ that there is a surjective $*$-homomorphism $A \rtimes_{\alpha,\mu|_N} N \to A \rtimes_{\alpha,\mathcal{E}^N_{\mathfrak{Corr}}} N$. Hence we get a chain of $*$-homomorphisms

$$\begin{aligned} A \rtimes_{\alpha,\mu|_N} N \to A \rtimes_{\alpha,\mathcal{E}^N_{\mathfrak{Corr}}} N \to (A \otimes C_0(G/N)) \rtimes_{\alpha,\mathcal{E}^N_{\mathfrak{Corr}}} N \\ \to \mathcal{M}((A \otimes C_0(G/N)) \rtimes_{\alpha\otimes\tau,\mu} G), \end{aligned} \tag{7.5}$$

in which the second homomorphism is given by functoriality for the generalised homomorphism $A \to \mathcal{M}(A \otimes C_0(G/N)); a \mapsto a \otimes 1$. It is then easy to check on the generators that the composition of the maps in (7.5) coincides with the faithful map (7.4). Hence the first morphism $A \rtimes_{\alpha,\mu|_N} N \to A \rtimes_{\alpha,\mathcal{E}^N_{\mathfrak{Corr}}} N$ in (7.5) must be faithful, too. Hence the result. □

8 Some Questions

There are actually many open questions related to the study of exotic crossed products and, in particular, the study of the minimal exact correspondence crossed-product functor $\rtimes_{\mathcal{E}^G_{\mathfrak{Corr}}}$. Many of those have been formulated and discussed in the papers [3] and [9] and we therefore want to restrict here to questions related to restriction, extension, and induction of functors from and to subgroups. The first question is quite obvious:

Question 8.1 Suppose H is a closed subgroup of the locally compact group G. Is it always true that the restriction $\rtimes_{\mathcal{E}^G_{\mathfrak{Corr}}|_H}$ of the minimal exact correspondence functor $\rtimes_{\mathcal{E}^G_{\mathfrak{Corr}}}$ for G is equal to the minimal exact correspondence functor $\rtimes_{\mathcal{E}^N_{\mathfrak{Corr}}}$ for H?

If the answer is "yes" then it follows from Proposition 6.6 that the validity of the reformulated version of the Baum-Connes conjecture due to Baum, Guentner, and Willett would pass from a group G to any of its closed subgroups. The previous section gives a positive answer if H is normal in G. At an earlier stage we thought that we have a positive answer at least for open subgroups H of G, but, unfortunately, we found a gap in our arguments. Any progress in this direction would be very appreciated by the authors.

To formulate our next question, we start with a lemma.

Lemma 8.2 *Suppose that $N \subseteq G$ is a closed normal subgroup of G and that $\alpha : G \to \mathrm{Aut}(A)$ is an action. Then there is an action $\gamma^{\mathcal{E}} : G \to \mathrm{Aut}(A \rtimes_{\alpha,\mathcal{E}^N} N)$ (with $\rtimes_{\mathcal{E}^N} := \rtimes_{\mathcal{E}^N_{\mathfrak{Corr}}}$) such that for each $g \in G$ the automorphism $\gamma^{\mathcal{E}}_g$ is given on the level of $C_c(N, A)$ by the formula*

$$(\gamma^{\mathcal{E}}_g(f))(n) = \delta_g \alpha_g(f(g^{-1}ng)),$$

where δ_g denotes the module for the automorphism $C_g(n) = gng^{-1}$ of N.

Proof It follows directly from Lemma 7.2 applied to the automorphism $\gamma_g = (\alpha_g, C_g)$ of (A, N, α) that the automorphisms $\gamma_g^{\mathscr{E}}$ exist for all $g \in G$ and one easily checks that $g \mapsto \gamma_g^{\mathscr{E}}$ is a homomorphism. Since for each $f \in C_c(N, A)$ the map $G \mapsto C_c(N, A); f \mapsto \gamma_g^{\mathscr{E}}(f)$ is continuous with respect to the inductive limit topology on $C_c(N, A)$, the action $\gamma^E : G \to \mathrm{Aut}(A \rtimes_{\alpha, \mathscr{E}^N} N)$ is strongly continuous. $\square$

Remark 8.3 Note that there are analogous actions $\gamma^{\mathrm{u}} : G \to \mathrm{Aut}(A \rtimes_{\alpha,\mathrm{u}} N)$ and $\gamma^{\mathrm{r}} : G \to \mathrm{Aut}(A \rtimes_{\alpha,\mathrm{r}} N)$ extending the action on $C_c(N, A)$ as in the above lemma. These actions are known as the *decomposition actions* of G on the respective crossed products by N. Indeed, if $i_N^{\mathrm{u}} : N \to U\mathscr{M}(A \rtimes_{\alpha,\mathrm{u}} N)$ and $i_N^{\mathrm{r}} : N \to U\mathscr{M}(A \rtimes_{\alpha,\mathrm{r}} N)$ are the canonical maps, then the pairs $(\gamma^{\mathrm{u}}, i_N^{\mathrm{u}})$ and $(\gamma^{\mathrm{r}}, i_N^{\mathrm{r}})$ are twisted actions in the sense of Green [16] of the pair (G, N) on $A \rtimes_{\alpha,\mathrm{u}} N$ and $A \rtimes_{\alpha,\mathrm{r}} N$, respectively, and we get canonical isomorphisms

$$(A \rtimes_{\alpha,\mathrm{u}} N) \rtimes_{(\gamma^{\mathrm{u}}, i_N^{\mathrm{u}}),\mathrm{u}} (G, N) \cong A \rtimes_{\alpha,\mathrm{u}} G \text{ and } (A \rtimes_{\alpha,\mathrm{r}} N) \rtimes_{(\gamma^{\mathrm{r}}, i_N^{\mathrm{u}}),\mathrm{r}} (G, N) \cong A \rtimes_{\alpha,\mathrm{r}} G$$

(see [16, Proposition 1] for the full crossed products and [29, Proposition 5.2] for the reduced case). Of course, if $i_N^{\mathscr{E}} : N \to U\mathscr{M}(A \rtimes_{\alpha, \mathscr{E}^N_{\mathfrak{Corr}}} N)$ denotes the canonical map, then $(\gamma^{\mathscr{E}}, i_N^{\mathscr{E}})$ is also a Green-twisted action of (G, N) on $A \rtimes_{\alpha, \mathscr{E}^N_{\mathfrak{Corr}}} N$ and we may wonder whether a similar decomposition isomorphism holds for the minimal exact correspondence functors. Note that by [13, Theorem 1] we know that every twisted action of (G, N) is Morita equivalent to an ordinary action of G/N. Using this it is not difficult to extend any strongly Morita compatible crossed-product functor for G/N to the category of twisted (G, N)-algebras (e.g., see [8, Definition 4.13]).

Thus it is natural to ask

Question 8.4 Suppose (A, α) is a G-algebra and N is a normal subgroup of G. Is there always a canonical decomposition isomorphism

$$(A \rtimes_{\alpha, \mathscr{E}^N_{\mathfrak{Corr}}} N) \rtimes_{\gamma^{\mathscr{E}}, i_N^{\mathscr{E}}, \mathscr{E}^{G/N}_{\mathfrak{Corr}}} (G, N) \xrightarrow{\sim} A \rtimes_{\alpha, \mathscr{E}^G_{\mathfrak{Corr}}} G?$$

Is it at least true if $G = N \times H$ is the direct product of two closed subgroups?

Unfortunately we did not succeed in proving such result even in the case where G is a direct product $N \times H$.

Note that the candidate for the isomorphism in Question 8.4 can be described on the level of functions with compact supports, but the precise description is a bit tedious and we refer to [16, Proposition 1] for the details. In case where $G = N \rtimes H$ is a semi-direct product group, the twisted action of $G/N \cong H$ can be replaced by an untwisted crossed product by H and the desired decomposition isomorphism should then be given on the level of compactly supported continuous functions by

$$\Phi : C_c(N \times H, A) \to C_c(H, C_c(N, A)); \ \Phi(f)(h) = f(\cdot, h). \tag{8.1}$$

We believe that positive answers to Questions 8.1 and 8.4 would give the main steps for extending the permanence results for the classical Baum-Connes conjecture as obtained in [10] to the reformulated conjecture in the sense of Baum, Guentner, and Willett. Note that we always have $\rtimes_{\mathcal{E}^N_{\mathfrak{Corr}}} = \rtimes_{\mathcal{E}^G_{\mathfrak{Corr}}|_N}$ by Theorem 7.13. If $G = N \times H$ it follows also that $\rtimes_{\mathcal{E}^H_{\mathfrak{Corr}}} = \rtimes_{\mathcal{E}^G_{\mathfrak{Corr}}|_H}$. Thus in this case the above question may be generalised to

Question 8.5 Suppose (A, α) is an $N \times H$-algebra and $\rtimes_\mu$ is a crossed-product functor for $G = N \times H$. Let $\Phi : C_c(N \times H, A) \to C_c(H, C_c(N, A))$ be given by $\Phi(f)(h) = f(\cdot, h)$. Does Φ always extend to a *-isomorphism

$$A \rtimes_{\alpha,\mu} (N \times H) \xrightarrow{\sim} (A \rtimes_{\alpha^N,\mu|_N} N) \rtimes_{\alpha^H,\mu|_H} H?$$

The answer is positive for the full and reduced crossed products but we know very little about the general case. Note that it follows from our Lemma 7.4 that for any crossed-product functor $\rtimes_\mu$ for G, any closed normal subgroup $N \subseteq G$, and any G-algebra (A, α) there is a twisted action $(\gamma^\mu, i_N^{\mu|_N})$ of (G, N) on $A \rtimes_{\alpha,\mu|_N} N$. Hence any crossed product functor $\rtimes_\nu$ for G/N allows a decomposition crossed product $(A \rtimes_{\alpha,\mu|_N} N) \rtimes_{(\gamma^\mu, i_N^{\mu|_N}),\nu} (G, N)$. Thus we may ask:

Question 8.6 Given a crossed-product functor $\rtimes_\mu$ for G is there always an associated crossed-product functor $\rtimes_\nu$ for G/N such that

$$(A \rtimes_{\alpha,\mu|_N} N) \rtimes_{(\gamma^\mu, i_N^{\mu|_N}),\nu} (G, N) \cong A \rtimes_{\alpha,\mu} G?$$

It is actually not clear to us how to relate crossed-product functors for G/N to crossed-product functors for G in a "canonical" way if G does not decompose as a semi-direct product $N \rtimes H$.

Acknowledgements Part of the work on this paper took place during a visit of the first and third authors to the Westfälische Wilhelms-Universität, Münster. We would like to thank this institution for its hospitality.

This work has been supported by Deutsche Forschungsgemeinschaft (SFB 878, Groups, Geometry & Actions), by CNPq/CAPES—Brazil, and by the US NSF (DMS 1401126).

References

1. R.J. Archbold, and J.S. Spielberg, *Topologically free actions and ideals in discrete C^*-dynamical systems*, Proc. Edinburgh Math. Soc. (2), **37** (1994), 119–124.
2. P. Baum, A. Connes, and N. Higson. *Classifying space for proper actions and K-theory of group C^*-algebras*, in C^*-Algebras: 1943–1993, San Antonio, TX, 1993, Contemp. Math. 167. Amer. Math. Soc., Providence, RI 1994. p. 240–291.
3. P. Baum, E. Guentner, and R. Willett, *Expanders, exact crossed products, and the Baum-Connes conjecture*, Ann. K-theory **1** (2015), 155–208.

4. N. P. Brown and E. Guentner, *New C^*-completions of discrete groups and related spaces*, Bull. Lond. Math. Soc., **45** (2013), 1181–1193.
5. N. P. Brown and N. Ozawa, C^*-algebras and finite-dimensional approximations, Graduate Studies in Mathematics 88, American Mathematical Society, Providence, RI, 2008, pages xvi+509.
6. A. Buss and S. Echterhoff, *Universal and exotic generalized fixed-point algebras for weakly proper actions and duality*, Indiana Univ. Math. J. **63** (2014), 1659–1701.
7. A. Buss and S. Echterhoff, *Imprimitivity theorems for weakly proper actions of locally compact groups*, Ergodic Theory Dynam. Systems **35** (2015), 2412–2457, doi 10.1017/etds.2014.36.
8. A. Buss and S. Echterhoff, *Maximality of dual coactions on sectional C^*-algebras of Fell bundles and applications*, Studia Math. **229** (2015), 233–262, doi 10.4064/sm8361-1-2016.
9. A. Buss, S. Echterhoff, and R. Willett, *Exotic crossed products and the Baum-Connes conjecture*, J. Reine Angew. Math. doi 10.1515/crelle-2015-0061 (prepublished electronically).
10. J. Chabert and S. Echterhoff, *Permanence properties of the Baum-Connes conjecture*, Doc. Math., **6** (2001), 127–183 (electronic).
11. J. Cuntz, *K-theoretic amenability for discrete groups*, J. Reine Angew. Math. **344** (1983), 180–195.
12. S. Echterhoff, *Crossed products, the Mackey-Rieffel-Green machine and applications*, eprint, arxiv 1006.4975v2, 2010.
13. S. Echterhoff, *Morita equivalent twisted actions and a new version of the Packer–Raeburn stabilization trick*, J. London Math. Soc. (2) **50** (1994), 170–186.
14. S. Echterhoff, S. P. Kaliszewski, J. Quigg, and I. Raeburn, *A categorical approach to imprimitivity theorems for C^*-dynamical systems*, Mem. Amer. Math. Soc. **180** (2006) no. 850, pp. viii+169.
15. P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France, **92** (1964), 181–236.
16. P. Green, Philip, *The local structure of twisted covariance algebras*, Acta Math. **140** (1978), 191–250.
17. M. Gromov, *Random walks in random groups*, Geom. Funct. Anal. **13** (2003), 73–146.
18. U. Haagerup, *An example of a non nuclear C^*-algebra that has the metric approximation property*, Invent. Math. **50** (1979), 289–293.
19. R. Howe and E.C. Tan, Non-abelian harmonic analysis, Universitext, Springer-Verlag, New York, NY, 1992, pp. xv+357.
20. N. Higson, V. Lafforgue, and G. Skandalis, *Counterexamples to the Baum-Connes conjecture*, Geom. Funct. Anal. **12** (2002), 330–354.
21. A. Hulanicki, *Means and Følner conditions on locally compact groups*, Studia Math. **27** (1966), 87–104.
22. P. Jolissaint, *Notes on C_0-representations and the Haagerup property*, Bull. Belg. Math. Soc. Simon Stevin **21** (2014), 263–274.
23. P. Julg and A. Valette, *K-theoretic amenability for* $\mathrm{SL}_2(\mathbf{Q}_p)$*, and the action on the associated tree*, J. Funct. Anal., **58** (1984), 194–215.
24. S. P. Kaliszewski, M. B. Landstad, and J. Quigg, *Exotic group C^*-algebras in noncommutative duality*, New York J. Math. **19** (2013), 689–711.
25. S. P. Kaliszewski, M. B. Landstad, and J. Quigg, *Exotic coactions*, preprint 2013, arxiv1305.5489.
26. S. P. Kaliszewski, M. B. Landstad, and J. Quigg, *Coaction functors*, preprint 2015, arxiv1505.03487.
27. D. Kerr, *C^*-algebras and topological dynamics: finite approximations and paradoxicality*, 2011, available on the author's website.
28. E. Kirchberg and S. Wassermann, *Exact groups and continuous bundles of C^*-algebras*, Math. Ann. **315** (1999), 169–203.
29. E. Kirchberg and S. Wassermann, *Permanence properties of C*-exact groups*, Doc. Math. **4** (1999), 513–558 (electronic).

30. R. Kunze and E. Stein, *Uniformly bounded representations and harmonic analysis of the* 2×2 *real unimodular group*, Amer. J. Math. **82** (1960), 1–62.
31. D. Miličić, *Topological representation of the group* C^**-algebra of* $SL(2, \mathbb{R})$, Glasnik Mat. Ser. III **6(26)** (1971), 231–246.
32. R. Okayasu, *Free group C*-algebras associated with* ℓ_p, Internat. J. Math. **25** (2014), 1450065 (12 pages).
33. D. Osajda, *Small cancellation labellings of some infinite graphs and applications*, eprint 2014, arxiv 1406.5015.
34. R. Scaramuzzi, *A notion of rank for unitary representations of general linear groups*, Trans. Amer. Math. Soc. **319** (1990), 349–379.
35. D. P. Williams, Crossed products of C^*-algebras, Mathematical Surveys and Monographs 134, American Mathematical Society, Providence, RI, 2007, pp. xvi+528.
36. M. Wiersma, L^p*-Fourier and Fourier-Stieltjes algebras for locally compact groups*, J. Funct. Anal. **269** (2015), 3928–3951.

On Hong and Szymański's Description of the Primitive-Ideal Space of a Graph Algebra

Toke M. Carlsen and Aidan Sims

Abstract In 2004, Hong and Szymański produced a complete description of the primitive-ideal space of the C^*-algebra of a directed graph. This article details a slightly different approach, in the simpler context of row-finite graphs with no sources, obtaining an explicit description of the ideal lattice of a graph algebra.

1 Introduction

The purpose of this paper is to present a new exposition, in a somewhat simpler setting, of Hong and Szymański's description of the primitive-ideal space of a graph C^*-algebra. Their analysis [7] relates the primitive ideals of $C^*(E)$ to the maximal tails T of E—subsets of the vertex set satisfying three elementary combinatorial conditions (see page 111). In previous work with Bates and Raeburn, Hong and Szymański had already studied the primitive ideals of $C^*(E)$ that are invariant for its gauge action. Specifically, [1, Theorem 4.7] shows that the gauge-invariant primitive ideals of $C^*(E)$ come in two flavours: those indexed by maximal tails in which every cycle has an entrance; and those indexed by *breaking vertices*, which receive infinitely many edges in E, but only finitely many in the maximal tail that they generate. Hong and Szymański completed this list by showing in [7, Theorem 2.10] that the non-gauge-invariant primitive ideals are indexed by pairs consisting of a maximal tail containing a cycle with no entrance, and a complex number of modulus 1.

The bulk of the work in [7] then went into the description of the Jacobson, or hull-kernel, topology on $\operatorname{Prim} C^*(E)$ in terms of the indexing set described in the preceding paragraph. Theorem 3.4 of [7] describes the closure of a subset of $\operatorname{Prim} C^*(E)$ in terms of the combinatorial data of maximal tails and breaking

T.M. Carlsen
Department of Science and Technology, University of the Faroe Islands, Nóatún 3, FO-100 Tórshavn, Faroe Islands
e-mail: toke.carlsen@gmail.com

A. Sims (✉)
School of Mathematics and Applied Statistics, University of Wollongong, NSW 2522, Australia
e-mail: asims@uow.edu.au

T.M. Carlsen et al. (eds.), *Operator Algebras and Applications*, Abel Symposia 12,
DOI 10.1007/978-3-319-39286-8_4

vertices, and the usual topology on the circle $\mathbb{T}$. (Gabe [6] subsequently pointed out and corrected a mistake in [7, Theorem 3.4], but there is no discrepancy for row-finite graphs with no sources.) The technical details and notation involved even in the statement of this theorem are formidable, with the upshot that applying Hong and Szymański's result requires discussion of a fair amount of background and notation. This is due to some extent to the complications introduced by infinite receivers in the graph (to see this, compare [7, Theorem 3.4] with the corresponding statement [7, Corollary 3.5] for row-finite graphs). But it is also caused in part by the numerous cases involved in describing how the different flavours of primitive ideals described in the preceding paragraph relate to one another topologically.

Here we restrict attention to the class of row-finite graphs with no sources originally considered in [2, 9, 10]; it is a well-known principal that results tend to be cleaner in this context. The C^*-algebra of an arbitrary graph E is a full corner of the C^*-algebra of a row-finite graph E_{ds} with no sources, called a Drinen–Tomforde desingularisation E [5], so in principal our results combined with the Rieffel correspondence can be used to describe the primitive-ideal space and the ideal lattice of any graph C^*-algebra. But in practice there is serious book-keeping hidden in this innocuous-sounding statement.

We take a somewhat different approach than Hong and Szymański. We start, as they do, by identifying all the primitive ideals (Theorem 3.7)—though we take a slightly different route to the result. Our next step is to state precisely when a given primitive ideal in our list belongs to the closure of some other set of primitive ideals (Theorem 4.1). We could then describe the closure operation along the lines of Hong and Szymanski's result, but here our approach diverges from theirs. We describe a list of (not necessarily primitive) ideals $J_{H,U}$ of $C^*(E)$ indexed by *ideal pairs*, consisting of a saturated hereditary set H and an assignment U of a proper open subset of the circle to every cycle with no entrance in the complement of H. We describe each $J_{H,U}$ concretely by providing a family of generators. We prove that the map $(H, U) \mapsto J_{H,U}$ is a bijection between ideal pairs and ideals, and describe the inverse assignment (Theorem 5.1). Finally, in Theorem 6.1, we describe the containment relation and the intersection and join operations on primitive ideals in terms of a partial ordering and a meet and a join operation on ideal pairs.

One can recover the closure of a subset $X \subseteq \operatorname{Prim} C^*(E)$, and so Hong and Szymański's result, either by using the characterisation of points in $\overline{X}$ from Theorem 4.1, or by computing $\bigcap X$ using Theorem 6.1 and listing all the primitive ideals that contain this intersection. To aid in doing the latter, we single out the ideal pairs that correspond to primitive ideals (Remark 5.3), and identify when a given $J_{H,U}$ is contained in a given primitive ideal (Lemma 5.2).

We hope that this presentation of the ideal structure of $C^*(E)$ when E is row-finite with no sources will provide a useful and gentle introduction to Hong and Szymański's beautiful result for arbitrary graphs; and in particular that it will be helpful to readers familiar with the usual listing of gauge-invariant ideals using saturated hereditary sets.

1.1 Background

We assume familiarity with Raeburn's monograph [12] and take most of our notation and conventions from there. We have made an effort not to assume any further background.

We deal with row-finite directed graphs E with no sources; these consist of countable sets E^0, E^1 and maps $r, s : E^1 \to E^0$ such that r is surjective and finite-to-one. A Cuntz–Krieger family consists of projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ such that $s_e^* s_e = p_{s(e)}$ and $p_v = \sum_{r(e)=v} s_e s_e^*$. We will use the convention where, for example, for $v \in E^0$ the notation vE^1 means $\{e \in E^1 : r(e) = v\}$. A path of length $n > 0$ is a string $\mu = e_1 \dots e_n$ of edges where $s(e_i) = r(e_{i+1})$, and E^n denotes the collection of paths of length n. We write E^* for the collection of all finite paths (including the vertices, regarded as paths of length 0), and set $vE^* := \{\mu \in E^* : r(\mu) = v\}$, $E^*w := \{\mu \in E^* : s(\mu) = w\}$ and $vE^*w = vE^* \cap E^*w$ when $v, w \in E^0$.

2 Infinite Paths and Maximal Tails

Our first order of business is to relate maximal tails in a graph with the shift-tail equivalence classes of infinite paths (see also [8]).

Recall that a *maximal tail* in E^0 is a set $T \subseteq E^0$ such that:

(T1) if $e \in E^1$ and $s(e) \in T$, then $r(e) \in T$;
(T2) if $v \in T$ then there is at least one $e \in vE^1$ such that $s(e) \in T$; and
(T3) if $v, w \in T$ then there exist $\mu \in vE^*$ and $\nu \in wE^*$ such that $s(\mu) = s(\nu) \in T$.

If T is a maximal tail, there is a subgraph ET of E with vertices T and edges $E^1T := \{e \in E^1 : s(e) \in T\}$.

An *infinite path* in E is a string $x = e_1e_2e_3\cdots$ of edges such that $s(e_i) = r(e_{i+1})$ for all i. We let $r(x) := r(e_1)$. Two infinite paths x and y are shift-tail equivalent if there exist $m, n \in \mathbb{N}$ such that

$$x_{i+m} = y_{i+n} \quad \text{for all } i \in \mathbb{N}.$$

This shift-tail equivalence is (as the name suggests) an equivalence relation, and we write $[x]$ for the equivalence class of an infinite path x.

Shift-tail equivalence classes $[x]$ of infinite paths correspond naturally to irreducible representations of $C^*(E)$ (see Lemma 3.2). However, the corresponding primitive ideals depend not on $[x]$, but only on the maximal tail consisting of vertices that are the range of an infinite path in $[x]$. The next lemma describes the relationship between shift-tail equivalence classes of infinite paths and maximal tails.

Lemma 2.1 *Let E be a row-finite graph with no sources. A set $T \subseteq E^0$ is a maximal tail if and only if there exists $x \in E^\infty$ such that $T = [x]^0 := \{r(y) : y \in [x]\}$.*

Proof First suppose that T is a maximal tail. List $T = (v_1, v_2, \dots)$. Set $\lambda_1 = \mu_1 = v_1 \in E^*$, and then inductively, having chosen $\mu_{i-1} \in v_{i-2}E^*$ and $\lambda_{i-1} \in v_{i-1}E^*$ with $s(\lambda_{i-1}) = s(\mu)$, use (T3) to find $\mu_i \in v_{i-1}E^*$ and $\lambda_i \in v_iE^*$ such that $s(\mu_i) = s(\lambda_i) \in T$. We obtain an infinite path $x = \mu_1\mu_2\mu_3\cdots$. Since each $\lambda_i\mu_{i+1}\mu_{i+2}\cdots$ belongs to $[x]$, we have $T \subseteq [x]^0$. For the reverse containment, observe that if $v \in [x]^0$, then there exists $y \in [x]$ such that $v = r(y_1)$. By definition of $[x]$ there are m, i such that $s(y_m) = s(\mu_i)$. Since $\mu_i \in T$, m applications of (T1) give $r(y_1) \in T$. $\square$

We divide the maximal tails in E into two sorts. Those which have a cycle with no entrance, and those which don't. The main point is that, as pointed out in [7], if T contains a cycle without an entrance, then it contains just one of them, and is completely determined by this cycle.

A *cycle* in a graph E is a path $\mu = \mu_1 \dots \mu_n \in E^*$ such that $r(\mu_1) = s(\mu_n)$ and $s(\mu_i) \neq s(\mu_j)$ whenever $i \neq j$. Each cycle μ determines an infinite path $\mu^\infty := \mu\mu\mu\cdots$ and hence a maximal tail $T_\mu := [\mu^\infty]^0$; it is straightforward to check that

$$T_\mu = \{r(\lambda) : \lambda \in E^* r(\mu)\}.$$

Given a cycle $\mu \in E^*$ and a subset A of E^0 that contains $\{r(\mu_i) : i \leq |\mu|\}$, we say that μ *is a cycle with no entrance in* A if $\{e \in r(\mu_i)E^1 : s(e) \in A\} = \{\mu_i\}$ for each $1 \leq i \leq |\mu|$.

Lemma 2.2 *Let E be a row-finite graph with no sources. Suppose that $T \subseteq E^0$ is a maximal tail. Then either*

a) *there is a cycle μ with no entrance in T such that $T = T_\mu$, and this μ is unique up to cyclic permutation of its edges; or*
b) *there is no cycle μ with no entrance in T.*

Proof Suppose that there is a cycle μ with no entrance in T. Lemma 2.1 implies that $T = [x]^0$ for some infinite path x. So there exists $y \in [x]$ such that $r(y) = r(\mu)$, and since shift-tail equivalence is an equivalence relation, we then have $T = [y]^0$. Since μ has no entrance in T, the only element of E^∞ lying entirely within T and with range $r(\mu)$ is μ^∞. So $y = \mu^\infty$, and $T = [\mu^\infty]^0 = T_\mu$.

If ν is another cycle with no entrance in $T = T_\mu$ then $r(\nu)E^*r(\mu) \neq \emptyset$, say $\lambda \in r(\nu)E^*r(\mu)$. Since ν has no entrance in T, we have $\lambda\mu = \nu_1^\infty \cdots \nu_k^\infty$ for some k. In particular $\nu_{k-|\mu|+1}^\infty \cdots \nu_k^\infty = \mu$, and we deduce that $\nu = \mu_i \cdots \mu_{|\mu|}\mu_1 \cdots \mu_{i-1}$, where $i \equiv k + 1 \pmod{|\mu|}$. $\square$

We call a maximal tail T satisfying (a) in Lemma 2.2 a *cyclic maximal tail* and write $\mathrm{Per}(T) := |\mu|$. We call a maximal tail T satisfying (b) in Lemma 2.2 a *aperiodic maximal tail*, and define $\mathrm{Per}(T) := 0$.

3 The Irreducible Representations

In this section, we show that every primitive ideal of $C^*(E)$ naturally determines a corresponding maximal tail, and then construct a family of irreducible representations of $C^*(E)$ associated to each maximal tail of E.

The following lemma constructs a maximal tail from each primitive ideal of $C^*(E)$. It was proved for arbitrary graphs in [1, Lemma 4.1] using the relationship between ideals and saturated hereditary sets established there and that primitive ideals of separable C^*-algebras are prime. Here we present instead the direct representation-theoretic argument of [3, Theorem 5.3]. Recall that a saturated hereditary subset of E^0 is a subset whose complement satisfies axioms (T1) and (T2) of a maximal tail.

Lemma 3.1 ([1, Lemma 4.1]) *Let E be a row-finite graph with no sources. If I is a primitive ideal of $C^*(E)$, then $T := \{v \in E^0 : p_v \notin I\}$ is a maximal tail of E.*

Proof The set of $v \in E^0$ such that $p_v \in I$ is a saturated hereditary set by [12, Lemma 4.5] (see also [2, Lemma 4.2]). So its complement T satisfies (T1) and (T2). To establish (T3), fix $v, w \in T$. Take an irreducible representation $\pi : C^*(E) \to \mathscr{B}(\mathscr{H})$ such that $\ker(\pi) = I$. Since $v \in T$, we have $p_v \notin I$, and so $\pi(p_v)\mathscr{H} \neq \{0\}$. Fix $\xi \in \pi(p_v)\mathscr{H}$ with $\|\xi\| = 1$. Since $p_w \notin I$, the space $\pi(p_w)\mathscr{H}$ is also a nontrivial subspace of $\mathscr{H}$. Since π is irreducible, ξ is cyclic for π, and so there exists $a \in C^*(E)$ such that $\pi(p_w)\pi(a)\xi = \pi(p_w a p_v)\xi$ is nonzero. In particular, we have $\pi(p_w a p_v) \neq 0$. Since $C^*(E) = \overline{\operatorname{span}}\{s_\mu s_\nu^* : s(\mu) = s(\nu)\}$, and since $p_w s_\mu s_\nu^* p_v \neq 0$ only if $r(\mu) = w$ and $r(\nu) = v$, we have

$$\pi(p_w a p_v) \in \overline{\operatorname{span}}\{\pi(s_\mu s_\nu^*) : r(\mu) = w, r(\nu) = v, s(\mu) = s(\nu)\} \setminus \{0\}.$$

So there exist $\mu, \nu \in E$ with $r(\mu) = w$, $r(\nu) = v$, $s(\mu) = s(\nu)$, and $\pi(s_\mu p_{s(\mu)} s_\nu^*) = \pi(s_\mu s_\nu^*) \neq 0$. In particular, $\pi(p_{s(\mu)}) \neq 0$, giving $p_{s(\mu)} \notin I$. So $s(\mu) \in T$ satisfies $wE^* s(\mu), vE^* s(\mu) \neq \emptyset$. □

Next we show how to recover a family of primitive ideals from the shift-tail equivalence class of an infinite path.

Lemma 3.2 *Let E be a row-finite directed graph with no sources. For $x \in E^\infty$ and $z \in \mathbb{T}$, there is an irreducible representation $\pi_{x,z} : C^*(E) \to \mathscr{B}(\ell^2([x]))$ such that for all $y \in [x]$, $v \in E^0$ and $e \in E^1$, we have*

$$\pi_{x,z}(p_v)\delta_y = \begin{cases} \delta_y & \text{if } r(y_1) = v \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \pi_{x,z}(s_e)\delta_y = \begin{cases} z\delta_{ey} & \text{if } r(y_1) = s(e) \\ 0 & \text{otherwise.} \end{cases}$$

We have $\{v \in E^0 : p_v \notin \ker(\pi_{x,z})\} = [x]^0$.

Proof It is easy to check that $\ell^2([x])$ is an invariant subspace of $\ell^2(E^\infty)$ for the infinite-path space representation of [12, Example 10.2] (with $k = 1$). So

the infinite-path space representation reduces to a representation on $\mathscr{B}(\ell^2([x]))$. Precomposing with the gauge automorphism $\gamma_z : s_e \mapsto zs_e$ of [12, Proposition 2.1] yields a representation $\pi_{x,z}$ satisfying the desired formula.

To see that $\pi_{x,z}$ is irreducible, first observe that for each x, the rank-1 projection $\theta_{x,x}$ onto $\mathbb{C}\delta_x$ is equal to the strong limit

$$\theta_{x,x} = \lim_{n\to\infty} \pi_{x,z}(s_{x_1\cdots x_n} s^*_{x_1\cdots x_n}).$$

If $y, z \in [x]$, then $y = \mu w$ and $z = \nu w$ for some $\mu, \nu \in E^*$ and $w \in [x]$. Thus the rank-1 operator $\theta_{y,z}$ from $\mathbb{C}\delta_z$ to $\mathbb{C}\delta_y$ is in the strong closure of the image of $\pi_{x,z}$:

$$\theta_{y,z} = z^{|\nu|-|\mu|}\pi_{x,z}(s_\mu)\theta_{w,w}\pi_{x,z}(s^*_\nu) = \lim_{n\to\infty} \pi_{x,z}(z^{|\nu|-|\mu|} s_{\mu w_1\cdots w_n} s^*_{\nu w_1\cdots w_n}).$$

So $\mathscr{K}(\ell^2([x]))$ is contained in the strong closure of $\pi_{x,z}(C^*(E))$. Thus $\pi_{x,z}$ is irreducible.

If $v \notin [x]^0$, then $v \neq r(y_1)$ for any $y \in [x]$, and so the formula for $\pi_{x,z}$ shows that $p_v \in \ker(\pi_{x,z})$. On the other hand, if $v \in [x]^0$, then we can find $y \in [x]$ with $r(y_1) = v$, and then $\pi_{x,z}(p_v)\delta_y = \delta_y \neq 0$. □

Next we want to know when two of the irreducible representations constructed as in Lemma 3.2 have the same kernel. For the following, recall that if $H \subseteq E^0$ is a hereditary set (i.e., $E^0 \setminus H$ satisfies axiom (T1) of a maximal tail.), then $E \setminus EH$ is the subgraph of E with vertices $E^0 \setminus H$ and edges $E^1 \setminus E^1 H$. Note that if T is a maximal tail, then $H := E^0 \setminus T$ is a saturated hereditary set, and then $E \setminus EH = ET$.

Proposition 3.3 *Let E be a row-finite graph with no sources. Fix $x, y \in E^\infty$ and $w, z \in \mathbb{T}$. The irreducible representations $\pi_{x,w}$ and $\pi_{y,z}$ have the same kernel if and only if $[x]^0 = [y]^0$ and $w^{\mathrm{Per}([x]^0)} = z^{\mathrm{Per}([x]^0)}$.*

The crux of the proof of Proposition 3.3 is Lemma 3.5, which we state separately because it is needed again later to prove that every primitive ideal is of the form $I_{\pi,z}$. Our proof of Lemma 3.5 in turn relies on the following standard fact about kernels of irreducible representations; we thank the anonymous referee for suggesting the following elementary proof.

Lemma 3.4 *Let A be a C^*-algebra, let J be an ideal of A, and let π_1 and π_2 be irreducible representations of A that do not vanish on J. Then $\ker(\pi_1) = \ker(\pi_2)$ if and only if $\ker(\pi_1) \cap J = \ker(\pi_2) \cap J$.*

Proof The “ $\Longrightarrow$ ” direction is obvious. Suppose that $\ker(\pi_1) \cap J = \ker(\pi_2) \cap J$. By symmetry, it suffices to show that $\ker(\pi_1) \subseteq \ker(\pi_2)$. Since π_2 is irreducible, $\ker(\pi_2)$ is primitive, and hence prime (see, for example, [11, Proposition 3.13.10]). By assumption, we have $\ker(\pi_1) \cap J = \ker(\pi_2) \cap J \subseteq \ker(\pi_2)$. Since π_2 does not vanish on J, we have $J \not\subseteq \ker(\pi_2)$. So primeness of $\ker(\pi_2)$ forces $\ker(\pi_1) \subseteq \ker(\pi_2)$. □

Lemma 3.5 *Let E be a row-finite graph with no sources, and suppose that T is a maximal tail of E. Let $H := E^0 \setminus T$.*

(1) Suppose that T is an aperiodic tail and π is an irreducible representation of $C^*(E)$ such that $\{v \in E^0 : \pi(p_v) \neq 0\} = T$. Then $\ker\pi$ is generated as an ideal by $\{p_v : v \in H\}$.

(2) Suppose that T is a cyclic tail and that μ is a cycle with no entrance in T. Suppose that π_1 and π_2 are irreducible representations of $C^*(E)$ such that

$$\{v : \pi_1(p_v) \neq 0\} = T = \{v : \pi_2(p_v) \neq 0\}.$$

Then each π_i restricts to a one-dimensional representation of $C^*(s_\mu)$, and $\ker\pi_1 = \ker\pi_2$ if and only if $\pi_1(s_\mu) = \pi_2(s_\mu)$ as complex numbers. Each $\ker\pi_i$ is generated as an ideal by $\{p_v : v \in H\} \cup \{\pi_i(s_\mu)p_{r(\mu)} - s_\mu\}$.

Proof We start with some setup that is needed for both statements. Let I be the ideal of $C^*(E)$ generated by $\{p_v : v \in H\}$. This H is a saturated hereditary set. If π is an irreducible representation such that $\{v \in E^0 : \pi(p_v) \neq 0\} = T$, then I is contained in $\ker\pi$ by definition. By [12, Remark 4.12], there is an isomorphism $C^*(E)/I \cong C^*(E \setminus EH)$ that carries $p_v + I$ to p_v for $v \in E^0 \setminus H$. Since $I \subseteq \ker\pi$, the representation π descends to an irreducible representation of $C^*(E)/I$, and hence determines an irreducible representation $\tilde{\pi}$ of $C^*(E \setminus EH)$ such that

$$\tilde{\pi}(p_v) = \pi(p_v) \quad \text{for } v \in E^0 \setminus H.$$

Now, for (1), if T is an aperiodic maximal tail, and π is as above, then every cycle in $E \setminus H$ has an entrance in $E \setminus H$, and $\tilde{\pi}$ is a representation of $C^*(E \setminus EH)$ such that $\tilde{\pi}(p_v) \neq 0$ for all $v \in (E \setminus EH)^0$. So the Cuntz–Krieger uniqueness theorem [12, Theorem 2.4] implies that $\tilde{\pi}$ is faithful. Hence $\ker\pi = I$, proving (1).

For (2), consider the ideal J of $C^*(E \setminus EH)$ generated by $p_{r(\mu)}$. Then $\tilde{\pi}_i(J) \neq \{0\}$ for $i = 1, 2$. So Lemma 3.4 implies that $\tilde{\pi}_1$ and $\tilde{\pi}_2$ have the same kernel if and only if $\ker(\tilde{\pi}_1) \cap J = \ker(\tilde{\pi}_2) \cap J$. Since J is generated as an ideal by $p_{r(\mu)}$, the corner $p_{r(\mu)}Jp_{r(\mu)} = \overline{\operatorname{span}}\{s_{\mu^n}s_{\mu^m}^* : m, n \in \mathbb{N}\}$ is full in J. Rieffel induction from a C^*-algebra to a full corner is implemented by restriction of representations [13, Proposition 3.24]. Since Rieffel induction carries irreducible representations to irreducible representations and induces a bijection between primitive-ideal spaces, we deduce that each $\tilde{\pi}_i$ is an irreducible representation of $C^*(s_\mu) \subseteq J$, and that

$$\ker\tilde{\pi}_1 = \ker\tilde{\pi}_2 \qquad \Longleftrightarrow \qquad \ker(\tilde{\pi}_1) \cap p_{r(\mu)}Jp_{r(\mu)} = \ker(\tilde{\pi}_2) \cap p_{r(\mu)}Jp_{r(\mu)}.$$

Since μ has no entrance, s_μ is a unitary element of $p_{r(\mu)}Jp_{r(\mu)}$, so $C^*(s_\mu) \cong C(\sigma(s_\mu))$. Since the irreducible representations of a commutative C^*-algebra are one-dimensional, we deduce that each $\tilde{\pi}_i$ is a one-dimensional representation of $C^*(s_\mu) \subseteq C^*(E \setminus EH)$ and hence each π_i is a one-dimensional representation of $C^*(s_\mu) \subseteq C^*(E)$. Moreover, $\tilde{\pi}_1$ and $\tilde{\pi}_2$ have the same kernel if and only if they are

implemented by evaluation at the same point z in $\sigma(s_\mu)$, and hence if and only if $\pi_1(s_\mu) = \pi_2(s_\mu)$.

For the final statement fix $i \in \{1, 2\}$. Since I is contained in the ideal J' generated by $\{p_v : v \in H\} \cup \{\pi_i(s_\mu)p_{r(\mu)} - s_\mu\}$, we have $J' = \ker \pi_i$ if and only if $\ker \tilde{\pi}_i$ is equal to the image J'' of J'/I in $C^*(E \setminus EH)$. Since Rieffel induction induces a bijection on ideal-spaces, $\ker \tilde{\pi}_i = J''$ if and only if $p_{r(\mu)} \ker \tilde{\pi}_i p_{r(\mu)} = p_{r(\mu)} J'' p_{r(\mu)}$. Both of these ideals coincide with the maximal ideal corresponding to the complex number $\pi_i(s_\mu) \in \sigma(s_\mu)$, so we are done. □

Proof of Proposition 3.3 Lemma 3.2 implies that if $\ker \pi_{x,w} = \ker \pi_{y,z}$, then $[x]^0 = [y]^0$. So it suffices to prove that if $[x]^0 = [y]^0$, then

$$\ker \pi_{x,w} = \ker \pi_{y,z} \text{ if and only if } w^{\mathrm{Per}([x]^0)} = z^{\mathrm{Per}([x]^0)}. \tag{3.1}$$

For this we consider two cases. First suppose that $[x]^0$ is an aperiodic maximal tail. Then Lemma 3.5(1) implies that each of $\ker \pi_{x,w}$ and $\ker \pi_{y,z}$ is generated by $\{p_v : v \notin T\}$, and in particular the two are equal. Also, $w^{\mathrm{Per}([x]^0)} = w^0 = 1 = z^0 = z^{\mathrm{Per}([x]^0)}$, so the equivalence (3.1) holds.

Now suppose that $[x]^0$ is cyclic, and let μ be a cycle with no entrance in $[x]^0$. We must show that $\ker \tilde{\pi}_{x,w} = \ker \tilde{\pi}_{y,z}$ if and only if $w^{|\mu|} = z^{|\mu|}$. Since μ has no entrance, both $\pi_{x,w}(p_{r(\mu)})\ell^2([x])$ and $\pi_{y,z}(p_{r(\mu)})\ell^2([y])$ are equal to the one-dimensional space $\mathbb{C}\delta_{\mu^\infty}$, and we have

$$\pi_{x,w}(s_\mu)\delta_{\mu^\infty} = w^{|\mu|}\delta_{\mu^\infty} \qquad \text{and} \qquad \pi_{y,z}(s_\mu)\delta_{\mu^\infty} = z^{|\mu|}\delta_{\mu^\infty}.$$

So, identifying the image of $\pi_{x,w}(C^*(s_\mu))$ with $\mathbb{C}$, we have $\pi_{x,w}(s_\mu) = w^{|\mu|}$ and similarly $\pi_{y,z}(s_\mu) = z^{|\mu|}$. So Lemma 3.5(2) shows that $\ker \pi_{x,w} = \ker \pi_{y,z}$ if and only if $z^{|\mu|} = w^{|\mu|}$. □

We are now ready for our first main result—a catalogue of the primitive ideals of $C^*(E)$. Proposition 3.3 says that the following definition makes sense.

Definition 3.6 Let E be a row-finite directed graph with no sources. Suppose that T is a maximal tail in E^0 and that $z \in \{w^{\mathrm{Per}(T)} : w \in \mathbb{T}\} \subseteq \mathbb{T}$. We define

$$I_{T,z} := \ker \pi_{x,w} \text{ for any } (x, w) \in E^\infty \times \mathbb{T} \text{ such that } [x]^0 = T \text{ and } w^{\mathrm{Per}(T)} = z.$$

Theorem 3.7 *The map* $(T, z) \mapsto I_{T,z}$ *is a bijection from*

$$\{(T, w^{\mathrm{Per}(T)}) : T \textit{ is a maximal tail, } w \in \mathbb{T}\}$$

to $\mathrm{Prim}\, C^*(E)$.

Proof Lemma 3.2 shows that each $I_{T,z}$ is a primitive ideal. Proposition 3.3 shows that $(T,z) \mapsto I_{T,z}$ is injective. So we just have to show that it is surjective. Fix a primitive ideal J of $C^*(E)$, let $T = \{v : p_v \notin J\}$, and let π be an irreducible representation of $C^*(E)$ with kernel J. Then T is a maximal tail according to Lemma 3.1. We must show that J has the form $I_{T,z}$.

If T is aperiodic, then Lemma 3.5(1) shows that $J = \ker\pi = \ker\pi_{x,1} = I_{[x]^0,1}$ for any x such that $[x]^0 = T$.

If T is cyclic, let μ be a cycle with no entrance in T. Lemma 3.5(2) shows that $\pi(C^*(s_\mu))$ is one-dimensional, so we can identify $\pi(s_\mu)$ with a nonzero complex number z. Since s_μ is an isometry, $|z| = 1$. Now Lemma 3.5(2) implies that any $w \in \mathbb{T}$ with $w^{|\mu|} = z$ satisfies $\ker\pi = \ker\pi_{[\mu^\infty]^0,w} = I_{[x]^0,z}$. □

4 The Closure Operation

The Jacobson, or hull-kernel, topology on $\operatorname{Prim} C^*(E)$ is the one determined by the closure operation $\overline{X} = \{I \in \operatorname{Prim} C^*(E) : \bigcap_{J\in X} J \subseteq I\}$. The ideals of $C^*(E)$ are in bijection with the closed subsets of $\operatorname{Prim} C^*(E)$: the ideal I_X corresponding to a closed subset X is

$$I_X := \bigcap\nolimits_{J\in X} J.$$

So the first step in describing the ideals of $C^*(E)$ is to say when a primitive ideal I belongs to the closure of a set X of primitive ideals. We do so with the following theorem.

Theorem 4.1 *Let E be a row-finite graph with no sources. Let X be a set of pairs (T,z) consisting of a maximal tail T and an element z of $\{w^{\operatorname{Per}(T)} : w \in \mathbb{T}\}$. Consider another such pair (S,w). Then $\bigcap_{(T,z)\in X} I_{T,z} \subseteq I_{S,w}$ if and only if both of the following hold:*

a) $S \subseteq \bigcup_{(T,z)\in X} T$, and
b) if S is a cyclic tail and the cycle μ with no entrance in S also has no entrance in $\bigcup_{(T,z)\in X} T$, then

$$w \in \overline{\{z : (S,z) \in X\}}.$$

We will need the following simple lemma in the proof Theorem 4.1, and at a number of other points later in the paper.

Lemma 4.2 *Let E be a row-finite graph with no sources, let H be a saturated hereditary subset of $C^*(E)$ and let μ be a cycle with no entrance in $E^0 \setminus H$. Let I_H be the ideal of $C^*(E)$ generated by $\{p_v : v \in H\}$. Then there is an isomorphism*

$$(p_{r(\mu)}C^*(E)p_{r(\mu)})/(p_{r(\mu)}I_H p_{r(\mu)}) \cong p_{r(\mu)}C^*(E \setminus EH)p_{r(\mu)}$$

carrying $s_\mu + p_{r(\mu)} I_H p_{r(\mu)}$ to s_μ, and there is an isomorphism of $p_{r(\mu)} C^*(E \setminus EH) p_{r(\mu)}$ onto $C(\mathbb{T})$ carrying s_μ to the generating monomial function $z \mapsto z$.

Proof Remark 4.12 of [12] shows that there is an isomorphism $C^*(E)/I_H \cong C^*(E \setminus EH)$ that carries $s_e + I_H$ to s_e if $e \in E^1 \setminus E^1 H$ and to zero otherwise. This restricts to the desired isomorphism $p_{r(\mu)} C^*(E) p_{r(\mu)} / p_{r(\mu)} I_H p_{r(\mu)} \cong p_{r(\mu)} C^*(E \setminus EH) p_{r(\mu)}$. The element $s_\mu \in p_{r(\mu)} C^*(E \setminus EH) p_{r(\mu)}$ satisfies $s_\mu^* s_\mu = p_{r(\mu)} = s_\mu s_\mu^*$ because μ has no entrance in $E^0 \setminus H$. So it suffices to show that the spectrum of s_μ calculated in $p_{r(\mu)} C^*(E \setminus EH) p_{r(\mu)}$ is $\mathbb{T}$. To see this, observe that the gauge action γ satisfies $\gamma_w(s_\mu) = w^{|\mu|}(s_\mu)$. So for $\lambda, w \in \mathbb{T}$, $\lambda p_{r(\mu)} - s_\mu$ is invertible if and only if $\gamma_w(\lambda p_{r(\mu)} - s_\mu) = w^{|\mu|}(w^{-|\mu|} \lambda p_{r(\mu)} - s_\mu)$ is invertible. That is, $\sigma(\mu)$ is invariant under rotation by elements of the form $w^{|\mu|}$, which comprise all of $\mathbb{T}$. Since the spectrum is nonempty, it follows that it is the whole circle. □

Proof of Theorem 4.1 We first prove the "if" direction. So suppose that (a) and (b) are satisfied. We consider two cases. First suppose that S is an aperiodic tail. Then $\mathrm{Per}(S) = \{0\}$, and so $w = 1$. For each maximal tail T of E, let

$$T_- := T \setminus \{v : v \text{ lies on a cycle with no entrance in } T\},$$

and let I_{T_-} be the ideal generated by $\{p_v : v \notin T_-\}$. If T is a cyclic maximal tail and μ is a cycle with no entrance in T, and if $z \in \{w^{\mathrm{Per}(T)} : w \in \mathbb{T}\}$, then Lemma 3.5(2) shows that $I_{T,z}$ is generated by $\{p_v : v \notin T\} \cup \{z p_{r(\mu)} - s_\mu\}$. So $I_{T,z} \subseteq I_{T_-}$. So it suffices to show that

$$\bigcap_{(T,z) \in X} I_{T_-} \subseteq I_{S,1}.$$

For this it suffices to show that $\bigcup_{(T,z) \in X} T_- \supseteq S$. We fix $v \in E^0 \setminus \bigcup_{(T,z) \in X} T_-$ and show that $v \notin S$. If $v \notin T$ for all $(T, z) \in X$, then it follows from (a) that $v \notin S$. So we may assume that $v \in \big(\bigcup_{(T,z) \in X} T\big) \setminus \big(\bigcup_{(T,Z) \in X} T_-\big)$. In particular, there exist pairs $(T, z) \in X$ such that $v \in T$. Fix any such pair. Since $v \notin T_-$, it must lie in a cycle μ in T with no entrance in T. Property (T1) shows that μ is contained entirely in T, and then Lemma 2.2 then gives $T = [\mu^\infty]^0 = r(E^* v)$. So μ has no entrance in $r(E^* v)$, and the only pairs $(T, z) \in X$ with $v \in T$ satisfy $T = r(E^* v)$. Thus μ has no entrance in $\bigcup_{(T,z) \in X} T$. Since $S \subseteq \bigcup_{(T,z) \in X} T$, and every cycle in S has an entrance in S, we deduce that μ does not lie in S and hence $v \notin S$ as required.

Now suppose that S is cyclic and μ is a cycle with no entrance in S. Let V be the set of vertices on μ. Lemma 2.2 gives $S = \{r(\alpha) : s(\alpha) \in V\}$. Since $S \subseteq \bigcup_{(T,z) \in X} T$, there exists $(T, z) \in X$ with $r(\mu) \in T$. Since T satisfies (T1), we deduce that the cycle μ lies in the subgraph ET of E. So there exists $(T, z) \in X$ such that $V \subseteq T$, and then $S \subseteq T$ because $S = \{r(\alpha) : s(\alpha) \in V\}$ and T satisfies (T1). So it suffices to show that

$$\bigcap_{(T,z) \in X, S \subseteq T} I_{T,z} \subseteq I_{S,\omega}.$$

For this, first suppose that there exists $(T, z) \in X$ such that T is a proper superset of S; say $v \in T \setminus S$. Since $S = \{r(\alpha) : s(\alpha) \in V\}$, we see that $vE^*V = \emptyset$, and hence $vE^*S = \emptyset$. So there exists $w \in T \setminus S$ such that VE^*w and vE^*w are both nonempty. Hence

$$T \supseteq \{r(\alpha) : s(\alpha) = w\} \supseteq \{r(\alpha) : s(\alpha) \in V\} = S.$$

If T is a cyclic tail, the cycle with no entrance that it contains lies outside of S, so the final statement of Lemma 3.5(2) shows that all the generators of $I_{T,z}$ belong to $I_{S,w}$; and if T is aperiodic, then all the generators of $I_{T,z}$ belong to $I_{S,w}$ by Lemma 3.5(1). In either case, we conclude that $I_{T,z} \subseteq I_{S,w}$, and hence $\bigcap_{(T,z)\in X, S\subseteq T} I_{T,z} \subseteq I_{S,\omega}$.

So it now suffices to show that $\bigcap_{z:(S,z)\in X} I_{S,z} \subseteq I_{S,w}$. Let I_S be the ideal generated by $\{p_v : v \notin S\}$. Then each $I_{S,z}$ contains I_S, as does $I_{S,w}$, so we need only show that in the quotient $C^*(E)/I_S \cong C^*(ES)$, the intersection of the images J_z of the $I_{S,z}$ is contained in J_w. Each J_z is generated by $zp_{r(\mu)} - s_\mu$ and is therefore contained in the ideal generated by $p_{r(\mu)}$, and similarly for J_w. Since the ideal generated by $p_{r(\mu)}$ is Morita equivalent to the corner determined by $p_{r(\mu)}$, it suffices to show that $\bigcap_{(S,z)\in X} p_{r(\mu)} J_z p_{r(\mu)} \subseteq p_{r(\mu)} J_w p_{r(\mu)}$. The isomorphism $p_{r(\mu)} C^*(ES) p_{r(\mu)} \cong C(\mathbb{T})$ of Lemma 4.2 carries each $p_{r(\mu)} J_z p_{r(\mu)}$ to $\{f \in C(\mathbb{T}) : f(z) = 0\}$. So $\bigcap_{(S,z)\in X} p_{r(\mu)} J_z p_{r(\mu)}$ is carried to

$$\left\{f \in C(\mathbb{T}) : f \equiv 0 \text{ on } \overline{\{z : (S,z) \in X\}}\right\},$$

and in particular is contained in the image of $p_{r(\mu)} J_w p_{r(\mu)}$.

We now prove the "only if" direction. To do this, we prove the contrapositive. So we first suppose that (a) does not hold. Then there is some $v \in S \setminus \bigcup_{(T,z)} T$. This implies that $p_v \in I_{(T,z)}$ for all (T, z), but $p_v \notin I_{S,w}$, and so $\bigcap_{(T,z)} I_{T,z} \nsubseteq I_{S,w}$ as required.

Now suppose that $S \subseteq \bigcup_{(T,z)} T$, that μ is a cycle with no entrance in S and that μ also has no entrance in $\bigcup_{(T,z)\in X} T$, and that $w \notin \overline{\{z : (S,z) \in X\}}$. As above, $S = \{r(\alpha) : s(\alpha) = r(\mu)\}$, and since μ has no entrance in any T, for each (T, z) we have either $T = S$ or $r(\mu) \notin T$. Whenever $r(\mu) \notin T$, we have $p_{r(\mu)} \in I_{(T,z)}$, and so $\bigcap_{(T,z)} p_{r(\mu)} I_{T,z} p_{r(\mu)} = \bigcap_{(S,z)\in X} p_{r(\mu)} I_{S,z} p_{r(\mu)}$. Once again taking quotients by I_S, it suffices to show that

$$\bigcap_{(S,z)\in X} p_{r(\mu)} J_z p_{r(\mu)} \nsubseteq p_{r(\mu)} J_w p_{r(\mu)}.$$

Since $w \notin \overline{\{z : (S,z) \in X\}}$, there exists $f \in C(\mathbb{T})$ such that $f(w) = 0$ and $f(z) = 1$ whenever $(S, z) \in X$. Let $g = 1 - f \in C(\mathbb{T})$. Then the images of the elements f and g belong to $\bigcap_{(S,z)\in X} p_{r(\mu)} J_z p_{r(\mu)}$ and $p_{r(\mu)} J_w p_{r(\mu)}$ respectively. Their sum is the

identity element $p_{r(\mu)}$, which does not belong to J_w. Thus

$$p_{r(\mu)}J_wp_{r(\mu)} + \bigcap_{(S,z)\in X} p_{r(\mu)}J_zp_{r(\mu)} \neq J_w.$$

Consequently, $\bigcap_{(S,z)\in X} p_{r(\mu)}J_zp_{r(\mu)} \nsubseteq p_{r(\mu)}J_wp_{r(\mu)}$. □

5 The Ideals of $C^*(E)$

We use Theorem 4.1 above to describe all the ideals of $C^*(E)$. We index them by what we call ideal pairs for E. To define these, given a saturated hereditary set H of E^0, we will write $\mathscr{C}(H)$ for the set

$$\mathscr{C}(H) := \{\mu : \mu \text{ is a cycle with no entrance in } E^0 \setminus H\}.$$

An *ideal pair* for E is then a pair (H, U) where H is a saturated hereditary set, and U is a function assigning to each $\mu \in \mathscr{C}(H)$ a proper open subset $U(\mu)$ of $\mathbb{T}$, with the property that $U(\mu) = U(\nu)$ whenever $[\mu^\infty] = [\nu^\infty]$.

Observe that if the maximal tail $E^0 \setminus H$ is aperiodic, so that $\mathscr{C}(H) = \emptyset$, then there is exactly one ideal pair of the form (H, U): the function U is the unique (trivial) function from the empty set to the collection of proper open subsets of $\mathbb{T}$.

To see how to obtain an ideal of $C^*(E)$ from an ideal pair, we need to do a little bit of background work.

For each open subset $U \subseteq \mathbb{T}$, we fix a function $h_U \in C(\mathbb{T})$ such that

$$\{z \in \mathbb{T} : h_U(z) \neq 0\} = U.$$

For example, we could take

$$h_U(z) := \inf\{|z - w| : w \notin U\}.$$

Let $\pi : C(\mathbb{T}) \to \ell^2(\mathbb{Z})$ be the faithful representation that carries the generating monomial $z \mapsto z$ to the bilateral shift operator $U : e_n \mapsto e_{n+1}$. The classical theory of Toeplitz operators says that if $P_+ : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{N})$ denotes the orthogonal projection onto the Hardy space $\overline{\operatorname{span}}\{e_n : n \geq 0\}$, then there is an isomorphism ρ from $P_+\pi(C(\mathbb{T}))P_+$ to the Toeplitz algebra $\mathscr{T} \subseteq \ell^2(\mathbb{N})$ generated by the unilateral shift operator S, such that if $q : \mathscr{T} \to C(\mathbb{T})$ is the quotient map that divides out the ideal of compact operators, then $q(\rho(P_+\pi(f)P_+)) = f$ for every $f \in C(\mathbb{T})$.

If $H \subseteq E^0$ is saturated and hereditary, then for each $\mu \in \mathscr{C}(H)$, we have $s_\mu s_\mu^* \leq p_{r(\mu)} = s_\mu^* s_\mu$, with equality precisely if μ has no entrance in E^0. So if μ has no entrance in E^0, then s_μ is unitary in $p_{r(\mu)}C^*(E)p_{r(\mu)}$, and we can apply the functional calculus in the corner to define a nonzero element $h_U(s_\mu) \in C^*(E)$. If μ has an

entrance in E^0, then $s_\mu s_\mu^* < s_\mu^* s_\mu$, so Coburn's theorem [4] gives an isomorphism $\psi : \mathscr{T} \cong C^*(s_\mu)$ that carries S to s_μ.

Using the preceding paragraph, given an ideal pair (H, U) and given $\mu \in \mathscr{C}(H)$, we obtain an element $\tau_\mu^U \in C^*(s_\mu) \subseteq p_{r(\mu)} C^*(E) p_{r(\mu)}$ given by

$$\tau_\mu^U := \begin{cases} h_{U(\mu)}(s_\mu) & \text{if } \mu \text{ has no entrance in } E^0 \\ \psi(\rho(P_+ \pi(h_{U(\mu)}) P_+)) & \text{otherwise.} \end{cases}$$

Theorem 5.1 *Let E be a row-finite graph with no sources. Let $\mathscr{I}_E$ denote the set of all ideal pairs for E. For each $(H, U) \in \mathscr{I}_E$, let $J_{H,U}$ be the ideal of $C^*(E)$ generated by*

$$\{p_v : v \in H\} \cup \{\tau_\mu^U : \mu \in \mathscr{C}(H)\}.$$

(1) The map $(H, U) \mapsto J_{H,U}$ is a bijection of $\mathscr{I}_E$ onto the collection of all closed 2-sided ideals of $C^*(E)$.

(2) Given an ideal I of $C^*(E)$, let $H_I := \{v \in E^0 : p_v \in I\}$, and for $\mu \in \mathscr{C}(H_I)$, let $U_I(\mu) = \mathbb{T} \setminus \operatorname{spec}_{(p_{r(\mu)}+I)(C^*(E)/I)(p_{r(\mu)}+I)}(s_\mu + I)$. Then (H_I, U_I) is an ideal pair and $I = J_{H_I,U_I}$.

Before proving the theorem, we need the following lemma.

Lemma 5.2 *Let E be a row-finite directed graph with no sources. Let (H, U) be an ideal pair for E, let T be a maximal tail of E and take $z \in \{w^{\mathrm{Per}(T)} : w \in \mathbb{T}\}$. Then $J_{H,U} \subseteq I_{T,z}$ if and only if both of the following hold:*

a) $H \subseteq E^0 \setminus T$; and

b) if T is cyclic and the cycle μ with no entrance in T belongs to $\mathscr{C}(H)$, then $z \notin U(\mu)$.

In particular, we have $\{v : p_v \in J_{H,U}\} = H$.

Proof For the "if" direction, fix $x \in E^\infty$ such that $T = [x]^0$ and $w \in \mathbb{T}$ such that $w^{\mathrm{Per}(T)} = z$. We just have to show that $\pi_{x,w}$ annihilates all the generators of $J_{H,U}$. For this, first fix $v \in H$. Then the final statement of Lemma 3.2 shows that $p_v \in \ker \pi_{x,w}$. Now fix $\mu \in \mathscr{C}(H)$. If $r(\mu) \notin T$, then $\pi_{x,w}(p_{r(\mu)}) = 0$ as above and then since $\tau_\mu^U \in p_{r(\mu)} C^*(E) p_{r(\mu)}$, it follows that $\pi_{x,w}(\tau_\mu^U) = 0$. So suppose that $r(\mu) \in T$. Since μ has no entrance in $E^0 \setminus H$ and since $T \subseteq E^0 \setminus H$, the cycle μ has no entrance in T. So T is a cyclic maximal tail, and $[x]^0 = [\mu^\infty]^0$ by Lemma 2.2. We then have $z \notin U(\mu)$ by hypothesis. The ideal I_H generated by $\{p_v : v \in H\}$ is contained in $\ker(\pi_{x,w})$, so $\pi_{x,w}$ descends to a representation $\tilde{\pi}_{x,w}$ of $C^*(E)/I_H$. Lemma 4.2 shows that $p_{r(\mu)} C^*(E) p_{r(\mu)} / p_{r(\mu)} I p_{r(\mu)} \cong C(\mathbb{T})$, and this isomorphism carries the restriction of $\tilde{\pi}_{x,w}$ to the one-dimensional representation ϵ_z given by evaluation at z. The isomorphism of Lemma 4.2 also carries $\tau_\mu^U + p_{r(\mu)} I p_{r(\mu)}$ to $h_{U(\mu)}$. Since $z \notin U(\mu)$, we have $\epsilon_z(h_{U(\mu)}) = 0$, and so $\pi_{x,w}(\tau_\mu^U) = 0$. So all of the generators of $J_{H,U}$ belong to $\ker \pi_{x,w}$ as required.

For the "only if" implication, we prove the contrapositive. Again fix $x \in E^\infty$ such that $T = [x]^0$ and $w \in \mathbb{T}$ such that $w^{\mathrm{Per}(T)} = z$, so that $I_{T,z} = \ker \pi_{x,w}$. First suppose that $H \not\subseteq E^0 \setminus T$; say $v \in T \cap H$. Then $p_v \in J_{H,U}$ by definition, but $p_v \notin \ker \pi_{x,w}$ by the final statement of Lemma 3.2, giving $J_{H,U} \not\subseteq \ker \pi_{x,w}$. Now suppose that $H \subseteq E^0 \setminus T$, that T is cyclic and that the cycle μ with no entrance in T belongs to $\mathscr{C}(H)$, but that $z \in U(\mu)$. Arguing as in the preceding paragraph, we see that $\pi_{x,w}(h_{U(\mu)}(z)p_{r(\mu)} - \tau_\mu^U) = 0$. Since $\tau_\mu^U \in J_{H,U}$, we deduce that $p_{r(\mu)} \in J_{H,U} + \ker \pi_{x,w}$. Since $p_{r(\mu)} \notin \ker \pi_{x,w}$ by Lemma 3.2, we deduce that $J_{H,U} \not\subseteq \ker \pi_{x,w}$.

For the final statement, observe that $H \subseteq \{v : p_v \in J_{H,U}\}$ by definition. For the reverse containment, recall that by definition of an ideal pair, each $U(\mu)$ is a proper subset of $\mathbb{T}$. So for each $\mu \in \mathscr{C}(H)$, we can choose $z_\mu \in \mathbb{T} \setminus U(\mu)$. By the preceding paragraphs, we have $J_{H,U} \subseteq I_{[\mu]^0, z_\mu}$ for each $\mu \in \mathscr{C}(H)$. For each $v \in E^0 \setminus H$ that does not belong to $[\mu^\infty]^0$ for any $\mu \in \mathscr{C}(H)$, we can choose an infinite path x^v in $E^0 \setminus H$ with $r(x_1^v) = v$. This $x^v \notin [\mu^\infty]$ for $\mu \in \mathscr{C}(H)$ because v does not belong to any $[\mu^\infty]^0$. So each $[x^v]^0$ is a maximal tail contained in the complement of H and the preceding paragraphs show that $J_{H,U} \subseteq I_{[x^v]^0,1}$. We now have

$$J_{H,U} \subseteq \Big(\bigcap\nolimits_\mu I_{[\mu]^0, z_\mu}\Big) \cap \Big(\bigcap\nolimits_v I_{[x^v]^0,1}\Big).$$

By construction, the right-hand side does not contain p_v for any $v \notin H$, and so we deduce that $v \notin H$ implies $p_v \notin J_{H,U}$ as required. □

Proof of Theorem 5.1 To prove the theorem, it suffices to show that the assignment $(H, U) \mapsto J_{H,U}$ is injective, and then prove statement (2).

The general theory of C^*-algebras says that every ideal of a C^*-algebra A is equal to the intersection of all of the primitive ideals that contain it. By definition, the topology on $\mathrm{Prim}(A)$ is the weakest one in which $\{I \in \mathrm{Prim}(A) : J \subseteq I\}$ is closed for every ideal J of A, and the map which sends J to this closed subset of $\mathrm{Prim}(A)$ is a bijection. So to prove that $(H, U) \mapsto J_{H,U}$ is injective, we just have to show that the closed sets $Y_{H,U} := \{I \in \mathrm{Prim}\, C^*(E) : J_{H,U} \subseteq I\}$ are distinct for distinct pairs (H, U).

By Lemma 5.2, we have

$$\begin{aligned} Y_{H,U} = \{I_{T,z} : T \subseteq E^0 \setminus H \text{ is a maximal tail, and}\\ &\text{if } T \text{ is cyclic and the cycle } \mu \text{ with no entrance in } T\\ &\text{also has no entrance in } H, \text{ then } z \notin U(\mu)\}. \end{aligned}$$

Suppose that (H_1, U_1) and (H_2, U_2) are distinct ideal pairs of E. We consider two cases. First suppose that $H_1 \neq H_2$. Without loss of generality, there exists $v \in H_1 \setminus H_2$. Since H_2 is saturated, there exists $e_1 \in vE^1$ such that $s(e_1) \notin H_2$. Since H_1 is hereditary, we have $s(e) \in H_1$. Repeating this argument we obtain edges $e_i \in s(e_{i-1})E^1$ with $s(e_i) \in H_1 \setminus H_2$, and hence an infinite path x lying in $(E \setminus EH_1) \setminus (E \setminus EH_2)$. Now $[x]^0$ is a maximal tail contained in $H_1 \setminus H_2$. If $[x]^0$ is

an aperiodic tail or is a cyclic tail such that the cycle with no entrance in $[x]^0$ has an entrance in $E \setminus EH_2$, we set $z = 1$. If $[x]^0 = [\mu^\infty]^0$ for some $\mu \in \mathscr{C}(H_2)$, we choose any $z \in \mathbb{T} \setminus U_2(\mu)$. Then Lemma 5.2 shows that $I_{[x]^0,z} \in Y_{H_2,U_2} \setminus Y_{H_1,U_1}$.

Now suppose that $H_1 = H_2$. Then $U_1 \neq U_2$, so we can find $\mu \in \mathscr{C}(H_1) = \mathscr{C}(H_2)$ such that $U_1(\mu) \neq U_2(\mu)$. Again without loss of generality, we can assume that there exists $z \in U_1(\mu) \setminus U_2(\mu)$, and then we have $I_{[\mu^\infty]^0,z} \in Y_{H_2,U_2} \setminus Y_{H_1,U_1}$. This completes the proof that the $Y_{H,U}$ are distinct.

It remains to prove (2). Given an ideal I, the set $H := H_I$ is a saturated hereditary set by [12, Lemma 4.5]. Since the ideal I_H generated by $\{p_v : v \in H\}$ is contained in I, Lemma 4.2 shows that $s_\mu + I$ is unitary in $(p_{r(\mu)} + I)C^*(E)/I(p_{r(\mu)} + I)$ for each $\mu \in C(H)$; so its spectrum is a closed subset of $\mathbb{T}$, showing that $U_I(\mu)$ is an open subset of $\mathbb{T}$. If $\mu, \nu \in C(H)$ with $[\mu^\infty] = [\nu^\infty]$, then $\mu^\infty = \alpha\nu^\infty$ for some initial segment α of μ^∞. The Cuntz–Krieger relations show that $s_\alpha^* s_\mu s_\alpha + I = s_\nu + I$ and $s_\alpha s_\nu s_\alpha^* + I = s_\mu + I$; so conjugation by $s_\alpha + I$ gives an isomorphism $C^*(s_\mu) + I \cong C^*(s_\nu) + I$, and in particular

$$\operatorname{spec}_{(p_{r(\mu)}+I)(C^*(E)/I)(p_{r(\mu)}+I)}(s_\mu + I) = \operatorname{spec}_{(p_{r(\nu)}+I)(C^*(E)/I)(p_{r(\nu)}+I)}(s_\nu + I),$$

giving $U_I(\mu) = U_I(\nu)$. So (H, U) is an ideal pair.

To see that $I = J_{H_I,U_I}$, we first check the containment $\supseteq$. For this, it suffices to show that every generator of J_{H_I,U_I} belongs to I. We have $p_v \in I$ for all $v \in H_I$ by definition. Fix $\mu \in \mathscr{C}(H_I)$; we must show that $\tau_\mu^{U_I} \in I$. For this, let I_H be the ideal of $C^*(E)$ generated by $\{p_v : v \in H\}$. Since I_H is contained in both I and J_{H_I,U_I} we just have to show that $J_{H_I,U_I}/I_H$ is contained in I/I_H. For this, let $\pi : p_{r(\mu)}C^*(E)p_{r(\mu)} \to C(\mathbb{T})$ be the composition of the isomorphism of Lemma 4.2 with the canonical surjection $p_{r(\mu)}C^*(E)p_{r(\mu)} \to (p_{r(\mu)} + I_H)(C^*(E)/I_H)(p_{r(\mu)} + I_H)$. Then $\pi(\tau_\mu^{U_I}) = h_{U_I(\mu)}$ vanishes on $\mathbb{T} \setminus U_I(\mu)$, which is $\operatorname{spec}_{(p_{r(\mu)}+I)(C^*(E)/I)(p_{r(\mu)}+I)}(s_\mu + I)$. Since the quotient map by the image of I under π is given by restriction of functions to $\operatorname{spec}_{(p_{r(\mu)}+I)(C^*(E)/I)(p_{r(\mu)}+I)}(s_\mu + I)$, it follows that $\tau_\mu^{U_I} + I_H \in I/I_H$ as required.

For the reverse containment, recall that every ideal of $C^*(E)$ is the intersection of the primitive ideals that contain it, so it suffices to show that if $I_{S,w} \in Y_{H_I,U_I}$, then $I \subseteq I_{S,w}$. Fix $I_{S,w} \in Y_{H_I,U_I}$. We can express I as an intersection of primitive ideals and therefore, by Theorem 3.7, we have $I = \bigcap_{(T,z)\in X} I_{T,z}$ for some set X of pairs consisting of a maximal tail T and an element $z \in \{u^{\operatorname{Per}(T)} : u \in \mathbb{T}\}$. We then have

$$v \in H_I \iff p_v \in I \iff p_v \in \textstyle\bigcap_{(T,z)\in X} I_{T,z} \iff v \in \bigcap_{(T,z)\in X} E^0 \setminus T,$$

and we deduce that $H_I = E^0 \setminus \bigcup_{(T,z)\in X} T$. Since $I_{S,w} \in Y_{H_I,U_I}$, we have $S \subseteq E^0 \setminus H_I = \bigcup_{(T,z)\in X} T$. So if S is an aperiodic tail, or is a cyclic tail such that the cycle μ with no entrance in S has an entrance in $\bigcup_{(T,z)\in X} T$, then Theorem 4.1 immediately gives $I = \bigcap_{(T,z)\in X} I_{T,z} \subseteq I_{S,w}$. So suppose that S is cyclic, and the cycle μ with no entrance in S has no entrance in $\bigcup_{(T,z)\in X} T$. Again using that $I_{S,w} \in Y_{H_I,U_I}$, we see that $w \notin U_I(\mu)$. Hence $w \in \operatorname{spec}_{(p_{r(\mu)}+I)(C^*(E)/I)(p_{r(\mu)}+I)}(s_\mu + I)$. So if π : $p_{r(\mu)}C^*(E)p_{r(\mu)} \to C(\mathbb{T})$ is the map described in the preceding paragraph, we have $f(w) = 0$ for all f in $\pi(I) = \bigcap_{(S,z)\in X} \pi(I_{S,z})$. Each $\pi(I_{S,z})$ is the set of functions

that vanishes at z, so we deduce that every function vanishing at every z for which $(S, z) \in X$ also vanishes at w; that is $w \in \overline{\{z : (S, z) \in X\}}$. Now Theorem 4.1 again gives $I = \bigcap_{(T,z)\in X} I_{T,z} \subseteq I_{S,w}$. □

Remark 5.3 To see where the primitive ideals of $C^*(E)$ fit into the catalogue of Theorem 5.1, first let us establish the convention that if $\mathscr{C}(H) = \emptyset$, then $\emptyset$ denotes the unique (trivial) function from $\mathscr{C}(H)$ to the collection of open subsets of $\mathbb{T}$, and that if $\mathscr{C}(H)$ is a singleton, then $\check{z}$ denotes the function on $\mathscr{C}(H)$ that assigns the value $\mathbb{T} \setminus \{z\}$ to the unique element of $\mathscr{C}(H)$. Now if T is a maximal tail and $z \in \{w^{\mathrm{Per}(T)} : w \in \mathbb{T}\}$, then Lemma 3.5 and the definition of the ideals $J_{H,U}$ show that

$$I_{T,z} = \begin{cases} J_{E^0\setminus T,\emptyset} & \text{if } T \text{ is aperiodic} \\ J_{E^0\setminus T,\check{z}} & \text{if } T \text{ is cyclic.} \end{cases}$$

Remark 5.4 The ideal $J_{H,U}$ is gauge invariant (i.e., $\gamma_z(J_{H,U}) = J_{H,U}$ for every $z \in \mathbb{T}$) if and only if $U(\mu) = \emptyset$ for every $\mu \in \mathscr{C}(H)$, in which case $J_{H,U} = I_H$. Thus, we recover from Theorem 5.1 the description of the gauge invariant ideals of $C^*(E)$ presented in [1, Theorem 4.1].

6 The Lattice Structure

To finish off the description of the lattice of ideals of $C^*(E)$, we describe the complete-lattice structure in terms of ideal pairs.

We define $\preceq$ on the set $\mathscr{I}_E$ of ideal pairs for a row-finite graph E with no sources by

$$(H_1, U_1) \preceq (H_2, U_2) \qquad \Longleftrightarrow \qquad H_1 \subseteq H_2 \text{ and } U_1(\mu) \subseteq U_2(\mu) \\ \text{for all } \mu \in \mathscr{C}(H_1) \cap \mathscr{C}(H_2).$$

In the following, given $X \subseteq \mathbb{T}$, we write $\mathrm{Int}(X)$ for the interior of X.

Theorem 6.1 *Let E be a row-finite graph with no sources.*

(1) Given ideal pairs (H_1, U_1) and (H_2, U_2) for E, we have $J_{H_1,U_1} \subseteq J_{H_2,U_2}$ if and only if $(H_1, U_1) \preceq (H_2, U_2)$.

(2) Given a set $K \subseteq \mathscr{I}_E$ of ideal pairs for E, we have $\bigcap_{(H,U)\in K} J_{H,U} = J_{H_K,U_K}$ where $H_K = \bigcap_{(H,U)\in K} H$, and $U_K(\mu) = \mathrm{Int}\big(\bigcap_{(H,U)\in K, \mu\in\mathscr{C}(H)} U(\mu)\big)$.

(3) Fix a set $K \subseteq \mathscr{I}_E$ of ideal pairs of E. Let A be the saturated hereditary closure of $\bigcup_{(H,U)\in K} H$. Let $B = \{r(\mu) : \mu \in \mathscr{C}(A) \text{ and } \bigcup_{(H,U)\in K,\mu\in\mathscr{C}(H)} U(\mu) = \mathbb{T}\}$. Let H^K be the saturated hereditary closure of $A \cup B$ in E^0, and for each $\mu \in \mathscr{C}(H^K)$, let $U^K(\mu) = \bigcup_{(H,U)\in K,\mu\in\mathscr{C}(H)} U(\mu)$. Then $\overline{\mathrm{span}}\big(\bigcup_{(H,U)\in K} J_{H,U}\big) = J_{H^K,U^K}$.

Proof (1): First suppose that $(H_1, U_1) \preceq (H_2, U_2)$. We show that every generator of J_{H_1,U_1} belongs to J_{H_2,U_2}. For each $v \in H_1$ we have $v \in H_2$ and therefore $p_v \in J_{H_2,U_2}$. Suppose that $\mu \in \mathscr{C}(H_1)$. If $r(\mu) \in H_2$, then $p_{r(\mu)} \in J_{H_2,U_2}$ and so $\tau_\mu^{U_1} \in p_{r(\mu)}C^*(E)p_{r(\mu)}$ belongs to J_{H_2,U_2} as well. So we may suppose that $r(\mu) \notin H_2$. Since $H_1 \subseteq H_2$ and since μ has no entrance in $E^0 \setminus H_1$, it cannot have an entrance in $E^0 \setminus H_2$, so it belongs to $\mathscr{C}(H_2)$. The ideal I_{H_1} generated by $\{p_v : v \in H_1\}$ is contained in both J_{H_1,U_1} and J_{H_2,U_2}. By Lemma 4.2, we have $(p_{r(\mu)} + I_{H_1})(C^*(E)/I_{H_1})(p_{r(\mu)} + I_{H_1}) \cong C(\mathbb{T})$ and this isomorphism carries $\tau_\mu^{U_1}$ to $h_{U_1(\mu)}$ and carries the image of J_{H_2,U_2} to $\{f \in C(\mathbb{T}) : f^{-1}(\mathbb{C} \setminus \{0\}) \subseteq U_2(\mu)\}$. Since $U_1(\mu) \subseteq U_2(\mu)$, it follows that the image of $\tau_\mu^{U_1}$ in the corner $(p_{r(\mu)} + I_{H_1})(C^*(E)/I_{H_1})(p_{r(\mu)} + I_{H_1})$ belongs to the image of J_{H_2,U_2}, and therefore $\tau_\mu^{U_1} + I_{H_1} \subseteq J_{H_2,U_2}$, giving $\tau_\mu^{U_1} \in J_{H_2,U_2}$.

Now suppose that $J_{H_1,U_1} \subseteq J_{H_2,U_2}$. The final statement of Lemma 5.2 shows that $H_1 \subseteq H_2$, so we must show that whenever $\mu \in \mathscr{C}(H_1) \cap \mathscr{C}(H_2)$, we have $U_1(\mu) \subseteq U_2(\mu)$. Theorem 5.1 (2) shows that

$$U_i(\mu) = \mathbb{T} \setminus \operatorname{spec}_{(p_{r(\mu)}+J_{H_i,U_i})(C^*(E)/J_{H_i,U_i})(p_{r(\mu)}+J_{H_i,U_i})}(s_\mu + J_{H_i,U_i}).$$

Since $J_{H_1,U_1} \subseteq J_{H_2,U_2}$, there is a homomorphism $q : C^*(E)/J_{H_1,U_1} \to C^*(E)/J_{H_2,U_2}$ that carries $s_\mu + J_{H_1,U_1}$ to $s_\mu + J_{H_2,U_2}$. In particular, q carries $p_{r(\mu)} + J_{H_1,U_1}$ to $p_{r(\mu)} + J_{H_2,U_2}$, and so induces a unital homomorphism between the corners determined by these projections. Since unital homomorphisms decrease spectra, we obtain

$$\begin{aligned}&\operatorname{spec}_{(p_{r(\mu)}+J_{H_2,U_2})(C^*(E)/J_{H_2,U_2})(p_{r(\mu)}+J_{H_2,U_2})}(s_\mu + J_{H_2,U_2})\\ &\qquad\subseteq \operatorname{spec}_{(p_{r(\mu)}+J_{H_1,U_1})(C^*(E)/J_{H_1,U_1})(p_{r(\mu)}+J_{H_1,U_1})}(s_\mu + J_{H_1,U_1}),\end{aligned}$$

and hence $U_1(\mu) \subseteq U_2(\mu)$.

(2): The ideal $\bigcap_{(H,U)\in K} J_{H,U}$ is the largest ideal that is contained in $J_{H,U}$ for every (H, U) in K. Since the map $(H, U) \to J_{H,U}$ is a bijection carrying $\preceq$ to $\subseteq$, it suffices to show that $(H_K, U_K) \preceq (H, U)$ for all $(H, U) \in K$, and is maximal with respect to $\preceq$ amongst pairs (H'', U'') satisfying $(H'', U'') \preceq (H, U)$ for all $(H, U) \in K$. The pair (H_K, U_K) satisfies $(H_K, U_K) \preceq (H, U)$ for all $(H, U) \in K$ by definition of H_K and U_K. Suppose that $(H'', U'') \preceq (H, U)$. Then $H'' \subseteq H$ for all $(H, U) \in K$, and hence $H'' \subseteq H_K$; and if $\mu \in \mathscr{C}(H'')\cap\mathscr{C}(H_K)$, and if $(H, U) \in K$ satisfies $\mu \in \mathscr{C}(H)$, then $U''(\mu) \subseteq U(\mu)$ because $(H'', U'') \preceq (H, U)$. So $U''(\mu)$ is an open subset of $\bigcap_{(H,U)\in K,\mu\in\mathscr{C}(H)} U(\mu)$, and therefore belongs to $\operatorname{Int}\big(\bigcap_{(H,U)\in K,\mu\in\mathscr{C}(H)} U(\mu)\big) = U_K$.

(3): The ideal $\overline{\operatorname{span}}\big(\bigcup_{(H,U)\in K} J_{H,U}\big)$ is the smallest ideal containing $J_{H,U}$ for every (H, U) in K. So as above it suffices to show that $(H, U) \preceq (H^K, U^K)$ for all $(H, U) \in K$, and that (H^K, U^K) is minimal with respect to $\preceq$ amongst pairs (H'', U'') satisfying $(H, U) \preceq (H'', U'')$ for all $(H, U) \in K$. The pair (H^K, U^K) satisfies $(H, U) \preceq (H^K, U^K)$ for all $(H, U) \in K$ by construction. Suppose that (H'', U'') is another ideal pair satisfying $(H, U) \preceq (H'', U'')$ for all $(H, U) \in K$. We just have to show that $(H^K, U^K) \preceq (H'', U'')$. We have $H \subseteq H''$ for every $(H, U) \in K$, and since H'' is saturated and hereditary, it follows that $A \subseteq H''$. If $v \in B$, then there

exists $\mu \in \mathscr{C}(A)$ such that $\bigcup_{(H,U)\in K, \mu\in\mathscr{C}(H)} U(\mu) = \mathbb{T}$, and then by compactness of $\mathbb{T}$, there are finitely many pairs $(H_1, U_1), \dots, (H_n, U_n) \in K$ such that $\mu \in \mathscr{C}(H_i)$ for each i, and $\bigcup_{i=1}^n U(\mu) = \mathbb{T}$. Choose a partition of unity $\{f_1, \dots, f_n\} \in C(\mathbb{T})$ subordinate to the U_i. Let I_A be the ideal of $C^*(E)$ generated by $\{p_v : v \in A\}$. Then each f_i belongs to the image of $p_{r(\mu)} J_{(H_i,U_i)} p_{r(\mu)}$ under the isomorphism of Lemma 4.2, and so $1 = \sum_i f_i$ belongs to the image of $\sum_{i=1}^n p_{r(\mu)} J_{H_i,U_i} p_{r(\mu)}$. Since each $(H_i, U_i) \preceq (H'', K'')$, it follows that 1 belongs to the image of $J_{(H'',K'')}$. But the preimage of 1 is $p_{r(\mu)} + I_A$, and we deduce that $p_{r(\mu)} \in J_{(H'',K'')}$. The final statement of Lemma 5.2 therefore implies that $v \in H''$. So $A \cup B \subseteq H''$, and since H'' is saturated and hereditary, it follows that $H^K \subseteq H''$. Now suppose that $\mu \in \mathscr{C}(H^K) \cap \mathscr{C}(H'')$. For each $z \in U^K(\mu)$, there exists $(H, U) \in K$ such that $\mu \in \mathscr{C}(H)$ and $z \in U(\mu)$. Since $(H, U) \preceq (H'', U'')$ and $\mu \in \mathscr{C}(H'') \cap \mathscr{C}(H)$, we deduce that $z \in U''(\mu)$. So $U^K(\mu) \subseteq U''(\mu)$. So we have $(H^K, U^K) \preceq (H'', U'')$ as required. $\square$

Acknowledgements The exposition of this paper has benefitted greatly from the suggestions of a very helpful referee. Thanks, whoever you are.

This research was supported by the Australian Research Council. Much of the work was completed between breathtaking hikes in the Faroe Islands; Aidan thanks Toke for his generous hospitality. Aidan also deeply thanks the organising committee of the 2015 Abel Symposium.

References

1. T. Bates, J.H. Hong, I. Raeburn and W. Szymański, *The ideal structure of the C^*-algebras of infinite graphs*, Illinois J. Math. **46** (2002), 1159–1176.
2. T. Bates, D. Pask, I. Raeburn and W. Szymański, *The C^*-algebras of row-finite graphs*, New York J. Math. **6** (2000), 307–324 (electronic).
3. T.M. Carlsen, S. Kang, J. Shotwell and A. Sims, *The primitive ideals of the Cuntz–Krieger algebra of a row-finite higher-rank graph with no sources*, J. Funct. Anal. **266** (2014), 2570–2589.
4. L.A. Coburn, *The C^*-algebra generated by an isometry*, Bull. Amer. Math. Soc. **73** (1967), 722–726.
5. D. Drinen and M. Tomforde, *The C^*-algebras of arbitrary graphs*, Rocky Mountain J. Math. **35** (2005), 105–135.
6. J. Gabe, *Graph C^*-algebras with a T_1 primitive ideal space*, Springer Proc. Math. Stat., 58, Operator algebra and dynamics, 141–156, Springer, Heidelberg, 2013.
7. J.H. Hong and W. Szymański, *The primitive ideal space of the C^*-algebras of infinite graphs*, J. Math. Soc. Japan **56** (2004), 45–64.
8. M. Ionescu, A. Kumjian, A. Sims and D.P. Williams, *A Stabilization Theorem for Fell Bundles over groupoids*, in preparation.
9. A. Kumjian, D. Pask and I. Raeburn, *Cuntz–Krieger algebras of directed graphs*, Pacific J. Math. **184** (1998), 161–174.
10. A. Kumjian, D. Pask, I. Raeburn and J. Renault, *Graphs, groupoids, and Cuntz–Krieger algebras*, J. Funct. Anal. **144** (1997), 505–541.
11. G.K. Pedersen, C^*-algebras and their automorphism groups, Academic Press, Inc., London-New York, 1979, ix+416.
12. I. Raeburn, Graph algebras, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2005, vi+113.
13. I. Raeburn and D.P. Williams, Morita equivalence and continuous-trace C^*-algebras, American Mathematical Society, Providence, RI, 1998, xiv+327.

Commutator Inequalities via Schur Products

Erik Christensen

Abstract For a self-adjoint unbounded operator D on a Hilbert space H, a bounded operator y on H and some Borel functions $g(t)$ we establish inequalities of the type

$$\|[g(D), y]\| \leq A_0\|y\| + A_1\|[D, y]\| + A_2\|[D, [D, y]]\| + \cdots + A_n\|[D, [D, \ldots [D, y] \ldots]]\|.$$

The proofs take place in a space of infinite matrices with operator entries, and in this setting it is possible to approximate the matrix associated to $[g(D), y]$ by the Schur product of a matrix approximating $[D, y]$ and a scalar matrix. A classical inequality on the norm of Schur products may then be applied to obtain the results.

1 Introduction

Let D be an unbounded self-adjoint operator on a Hilbert space H Then for a bounded operator y on H the commutator $[D, y]$ may be of interest for several reasons. From our personal point of view, we have previously studied *mathematical physics* [10], *noncommutative geometry* [5] and *operator algebras* [7] and found that commutators do occur for natural reasons. The most common one being that the expression $[D, y] := Dy - yD$ denotes the derivative of the function $t \to -ie^{itD}ye^{-itD}$ at $t = 0$. It is well known [1], that this derivative may or may not exist, and if it exists, it may be as a limit of difference quotients taken in the norm topology or as a limit in the ultraweak topology on $B(H)$, and it follows from [13] and [9] that the ultraweak limit may exist in cases where the uniform limit does not exist. If the ultraweak limit exists, we say that y is weakly-D differentiable and the bounded operator which is the limit is denoted $\delta_D(y)$, and if no mistakes are possible we will just write $\delta(y)$. Following [1] we let $C^1(D)$ denote the algebra of weakly D-differentiable operators on H, and in the article [3] we gave a presentation of some of the equivalent formulations of weak D-differentiability. Our personal contribution in this direction is based on a study of a space of infinite matrices with bounded

E. Christensen (✉)
Institute for Mathematics, Universitetsparken 5, 2100 Copenhagen O, Denmark
e-mail: echris@math.ku.dk

T.M. Carlsen et al. (eds.), *Operator Algebras and Applications*, Abel Symposia 12,
DOI 10.1007/978-3-319-39286-8_5

operators as entries, and we showed that for any bounded operator y the matrix associated with $[D, y]$ is always defined in this set up, but it represents a bounded operator exactly when y is weakly D-differentiable. This set up makes it possible to define all the algebraic operations needed to work with higher commutators, although the higher commutators may not exist as densely defined operators. In the article [4] we continued this work by studying higher commutators such as $[D, [D, \dots, [D, y] \dots]]$ and we showed—among other things—that an operator y is k times weakly D-differentiable if and only if all the matrix commutators $[D, [D, \dots [D, y] \dots]]$ up to the order k represent bounded operators.

Based on a question coming from *noncommutative geometry* and a question coming from the study of *perturbations of operator algebras* we got interested in relations between different commutators applied to the same bounded operator y. To be more precise, the question from non commutative geometry asks for relations between the commutators $[|D|, y|$ and $[D, y]$, where we define for D its numerical value $|D|$ by $|D| := (D^2)^{(1/2)}$. We show that for any natural number n and a bounded operator y on H, that if y is $n+1$ times weakly D-differentiable then it is n times weakly $|D|-$differentiable, and we obtain a norm estimate for the norm of the n'th derivative of y with respect to $|D|$ expressed in terms of norms of the $n+1$ first derivatives of y with respect to D. These results are presented in Sect. 5.

The question raised by the perturbation of operator algebras dealt with the question whether for a positive operator D, *almost commutation* with a bounded operator y is inherited by the square root $D^{(1/2)}$. It turned out that some results of this type are known, see for instance [12], so even though the set up we use in this article can be used to obtain some results of this type, we cannot produce inequalities as sharp as the known ones, so our results of this sort are not included in this article.

It turns out that for some complex Borel functions $g(t)$ on $\mathbb{R}$ the relations between the commutators $[g(D), y]$ and $[D, y]$ may, approximately, be expressed as a Schur product, and this is the basic observation which is behind the results in this article, as described in Sect. 3.

The Schur product between scalar matrices $A = (a_{ij})$ and $B = (b_{ij})$ is simply the element-wise product $(a_{ij}) * (b_{ij}) = (a_{ij}b_{ij})$ of the two matrices. With the kind help of Vern Paulsen we found a result by Bennett [2] on the norm of a Schur product of two scalar matrices, which we can use in our computations, and in Sect. 2 we have modified Bennett's result from the setting of scalar matrices to matrices of operators and in this way we have found solutions to the problems mentioned above and also to some general results of the type

$$\|[g(D), y]\| \le A_0\|y\| + A_1\|[D, y]\| + \dots + A_n\|[D, \dots [D, y] \dots]\|,$$

which hold, when $g(t)$ is a Borel function such that

$$\exists \alpha > 0,\, A \ge 0,\, B \ge 0\, \forall s, t \in \mathbb{R} : \quad |g(s) - g(t)| \le A + B|s - t|^{\alpha}.$$

This applies for instance to the cases when $g(t)$ is bounded or $g(t)$ is Hölder continuous or $g(t)$ is absolutely continuous with a derivative which may be written as a sum of an integrable function and an essentially bounded function. We present our abstract results in Sect. 5, where we also give applications to the cases where $g(t) = \log(t)$ and $g(t) = |t|$.

2 Schur Products, Row- and Column-Bounded Matrices

This section contains a single result, which is an *operator space version* [6, 15] of a result by Bennett, [2] Theorem 1.1 point (i). Given a couple of scalar matrices $A = (a_{ij})_{(i,j)\in I\times J}$ and $B = (b_{ij})_{(i,j)\in I\times J}$ over the same pair of sets of indices I and J, then the Schur product $C := A * B$ is defined as the matrix, of the same size, formed as the products of the entries, $c_{ij} := a_{ij}b_{ij}$. Bennett thought of such matrices over the scalars as operators between spaces $\ell^p(J)$ and $\ell^q(I)$. In particular he proved that if the matrix B corresponds to a bounded operator in $B(\ell^2(J), \ell^\infty(I))$ and the matrix A corresponds to a bounded operator in $B(\ell^1(J), \ell^2(I))$, then $A * B$ is a bounded operator in $B(\ell^2(J), \ell^2(I))$. The theory of Schur multipliers has been extended to the more general setting of operator spaces and operator modules, see for instance Paulsen's book [11] and the books by Pisier [14] and [15]. A closer look at the definition of the Schur product—$c_{ij} = a_{ij}b_{ij}$—immediately tells that the product may be meaningful in many different situations, such as the case when all the elements a_{ij} and b_{ij} are operators on the same Hilbert space, but also in the quite unrelated situation where the elements a_{ij} are scalars and the b_{ij} are elements in a family B_{ij} of Banach spaces. We are sure that we have not found all the possible generalizations of Bennett's theorem to the setting of operator spaces, but we have searched and found two versions, which are useful for this article and for our general interests.

The language we use is based on the words *column- and row-bounded* matrices of operators, which we define below.

The first version is formed as a statement on infinite matrices over the algebra of bounded operators algebras on a Hilbert space H. This result will not be used directly in this article, but we find quite easily a corollary, which we will use several times, in the sections to come.

Definition 2.1 Let J be an index set and H a Hilbert space, the set of all matrices $S = (s_{ij})_{(i,j\in J)}$ with entries $s_{ij} \in B(H)$ is denoted $M_J(B(H))$.

(i)

A matrix $S = (s_{ij})$ in $M_J(B(H))$ is said to be bounded if it represents a bounded operator in $B(H \otimes \ell^2(J))$. If S is bounded, the term $\|S\|$ means that norm.

(ii)

A matrix $S = (s_{ij})$ in $M_J(B(H))$ is said to be row bounded if there exists a $K \geq 0$ such that

$$\forall i \in J \, \forall J_0 \text{ finite } \subseteq J : \, \| \sum_{j \in J_0} s_{ij} s_{ij}^* \| \leq K^2.$$

The row norm $\|S\|_r$ is defined as the least possible such K.

(iii)

A matrix $S = (s_{ij})$ in $M_J(B(H))$ is said to be column bounded if there exists a $K \geq 0$ such that

$$\forall j \in J \, \forall J_0 \text{ finite } \subseteq J : \, \| \sum_{i \in J_0} s_{ij}^* s_{ij} \| \leq K^2.$$

The column norm $\|S\|_c$ is defined as the least possible such K.

(iv)

Let $K = (k_{ij})$ and $L = (l_{ij})$ be matrices in $M_J(B(H))$. Then their Schur product is the matrix $K * L$ in $M_J(B(H))$ defined by $(K * L)_{ij} := k_{ij} l_{ij}$.

In this setting our version of Bennett's result becomes

Theorem 2.2 *Let $K = (k_{ij})$ and $L = (l_{ij})$ be matrices in $M_J(B(H))$ such that K is row bounded and L is column bounded. Then $K * L$ is bounded in $M_J(B(H))$ and $\|K * L\| \leq \|K\|_r \|L\|_c$.*

Proof Let $J_0 \subseteq J$ be a finite set and $\xi = (\xi_j)_{(j \in J_0)}$ be in $\ell^2(J_0, H)$, then we will estimate the sum of non negative terms

$$\sum_{i \in J} \| \sum_{j \in J_0} k_{ij} l_{ij} \xi_j \|^2$$

by first fixing i and use the row boundedness of K to see that

$$\| \sum_{j \in J_0} k_{ij} l_{ij} \xi_j \|^2 \leq \|K\|_r^2 \sum_{j \in J_0} \|l_{ij} \xi_j\|^2.$$

Since we are summing non-negative reals, we may change the order of summation so we get from the last inequality, when we do the summation over i first, the following inequality

$$\sum_{i \in J} \| \sum_{j \in J_0} k_{ij} l_{ij} \xi_j \|^2 \leq \|K\|_r^2 \sum_{j \in J_0} \sum_{i \in J} \|l_{ij} \xi_j\|^2 \leq \|K\|_r^2 \|L\|_c^2 \sum_{j \in J_0} \|\xi_j\|^2 = \|K\|_r^2 \|L\|_c^2 \|\xi\|^2,$$

and the theorem follows. □

Remark 2.3 Matrices may be indexed by different indices for columns and rows, say I and J. The theorem above applies to this situation too, since such matrices embed naturally into *square matrices* over the set $I \cup J$.

The proof above applies to a situation which looks quite different. We will now look at a Hilbert space H and a family $(e_j)_{(j\in J)}$ of pairwise orthogonal projections in $B(H)$ with strong operator sum $I_{B(H)}$. Then we define $\mathscr{M}$ as the linear space of all matrices $A = (a_{ij})_{(i,j\in J)}$ and $a_{ij} \in e_iB(H)e_j$. The space $\mathscr{S}$ is defined as all the square scalar matrices over the index set J. A matrix $S = (s_{ij})$ in $\mathscr{S}$ is said to be bounded if it is the matrix of a bounded operator on $\ell^2(J, \mathbb{C})$, and similarly a matrix $A = (a_{ij})$ in $\mathscr{M}$ is bounded if there exists a bounded operator y on H, such that for each pair i, j we have $a_{ij} = e_iye_j$. The definitions of row- and column boundedness of S and A are made in the obvious way. Given a matrix $S = (s_{ij})$ in $\mathscr{S}$ and a matrix $A = (a_{ij})$ in $\mathscr{M}$, their Schur product $S * A = (s_{ij}a_{ij})$ is a matrix in $\mathscr{M}$ and we get a new version of Bennett's result

Corollary 2.4 *Let $S = (s_{ij})$ and $A = (a_{ij})$ be matrices in $\mathscr{S}$ and $\mathscr{M}$ such that S is row bounded and A is column bounded, then $S * A$ is bounded and $\|S * A\| \leq \|S\|_r\|A\|_c$.*

Proof The proof is the same as that of Theorem 2.2, even though the meaning of the product $s_{ij}a_{ij}$ is not identical to the meaning of the product $k_{ij}l_{ij}$ which appears in the previous proof.

It is also possible to apply the theorem directly and think of both S and A as elements in $M_J(B(H))$. □

3 The Basic Example

The results we present in this article are based on some well known aspects of elementary harmonic analysis. The unit circle $\mathbb{T}$ is equipped with the Hilbert space $H := L^2(\mathbb{T}, (1/2\pi)d\theta)$ based on the Haar probability measure, and the self-adjoint operator $D := (1/i)\frac{d}{d\theta}$ is just differentiation with respect to arc length, multiplied by $-i$. Basic Fourier analysis tells us that H has an orthonormal basis $e_n := e^{in\theta}$ of eigenvectors for D and with respect to this basis D has a matrix representation as a diagonal matrix with all the integers as the diagonal elements. Our basic observation is that if y is a bounded operator on H with matrix (y_{ij}) with respect to the basis (e_n), then the commutator $[D, y]$ makes sense as an infinite matrix even if there is no densely defined operator associated to this expression. Further, the commutator may be described as a Schur product of the matrix (y_{ij}) by the matrix $S = (S_{ij})$ given as $S_{ij} = (i - j)$. If g is a complex function on $\mathbb{Z}$, then we see that if we define a matrix

$$S := (S_{ij}) \quad S_{ij} := \begin{cases} 0 & \text{if } i = j \\ \frac{g(i)-g(j)}{i-j} & \text{if } i \neq j \end{cases} ,$$

then the commutator $[g(D), y]$ has a matrix which is given as the Schur product $(S_{ij}) * ([D, y]_{ij})$. The next sections will show some examples on how this basic relation may be transformed, so that it works for a general self-adjoint operator D, which may have a non-trivial continuous spectrum.

4 The Space $\mathscr{M}_D$ of Infinite Matrices Associated with D

Let D be a self-adjoint operator on a Hilbert space H and y a bounded operator on H, then we follow [1], and say that y is n times weakly D-differentiable, if for any pair of vectors ξ, η in H the function

$$\mathbb{R} \ni t \to \langle e^{itD} y e^{-itD} \xi, \eta \rangle$$

is n times differentiable. In the articles [3] and [4] we presented several equivalent properties to being n-times weakly D-differentiable and pointed out that most of these may be found in the text book [1], however one point of view was original, and we will recover the set up for this property here in order to set the stage for the arguments to come. If y is weakly D-differentiable, then we introduced the weak D-derivative $\delta(y)$ in the introduction, and we quote from [4], that if y is weakly D-differentiable, then

$$\forall \xi,\ \eta \in H: \quad \frac{d}{dt} \langle e^{itD} y e^{-itD} \xi, \eta \rangle = \langle e^{itD} \delta(y) e^{-itD} \xi, \eta \rangle$$

$$\mathrm{dom}([D, y]) = \mathrm{dom}(D) \quad \text{and} \quad \delta(y)\big|\mathrm{dom}(D) = i[D, y].$$

For an arbitrary bounded operator y the expression $i[D, y]$ is an operator on H, but it may have a small domain of definition, and it may not even be densely defined. In the basic example in Sect. 3 the commutator $i[D, y]$ always exists as an infinite matrix, and we will now present our way of transforming this *set up* to the case where the spectrum of D has a continuous part.

Given the self-adjoint operator D we define a sequence of spectral projections e_n for D, indexed over the integers $\mathbb{Z}$, by defining e_n as the spectral projection for D corresponding to the interval $[n - (1/2), n + (1/2)[$. Many of these projections may vanish, but this will have no effect on the computations to come. Then we define $\mathscr{M}_D$ as a space of matrices where the entries are bounded operators by

$$\mathscr{M}_D := \{(x_{ij}) \ : \ i, j \in \mathbb{Z}, \text{ and } x_{ij} \in e_i B(H) e_j\}.$$

Any bounded operator y on H has a natural representative $m(y)$ in $\mathscr{M}_D$ which is given by

$$m(y)_{ij} := e_i y e_j.$$

For each integer j the operator $d_j := De_j$ is bounded, defined on all of H and an element in $e_jB(H)e_j$, so we can define an element $m(D)$ in $\mathscr{M}_D$ by

$$m(D)_{ij} := \begin{cases} 0 \text{ if } i \neq j \\ d_j \text{ if } i = j. \end{cases}$$

For any element x in $\mathscr{M}_D$ the commutator $[m(D), x]$ is a well defined element in $\mathscr{M}_D$ which we denote $d(x) = \big(d(x)_{ij}\big)$ and it is given by

$$d(x)_{ij} := [m(D), x]_{ij} := d_i x_{ij} - x_{ij} d_j.$$

In this way, any iterated commutator such as $d^k(x) = [m(D), [m(D), \ldots [m(D), x] \ldots]]$ is meaningful and we showed in [4] that a bounded operator y on H is n times weakly D-differentiable if and only if all the iterated commutators $d^k\big(m(y)\big)$ for $1 \leq k \leq n$ represent bounded operators on H.

It is—of course—still just as difficult to prove boundedness for matrices as to establish boundedness on a given domain of vectors, but the matrix language makes it possible to assign a meaning to the expression $[D, y]$ in situations where the operator theoretical meaning is not existing. Further, the mapping $d : \mathscr{M}_D \to \mathscr{M}_D$ is clearly linear even if there may exist bounded operators y and z on H such that $[D, y + z] \neq [D, y] + [D, z]$.

There is still another advantage in working with $\mathscr{M}_D$, namely that $\mathscr{M}_D$ is left invariant by Schur multiplication by scalar matrices $S = (S_{ij})_{(i,j\in\mathbb{Z})}$. We realized this phenomenon, when working with the unit circle $\mathbb{T}$ as described in Sect. 3. In that section we describe how the matrix of a commutator $[g(D), y]$ may be obtained as a Schur product of a scalar matrix (S_{ij}) and the matrix for the commutator $[D, y]$. It is not possible to lift this simple identity to the case where D has continuous spectrum, since

$$[m(D), m(y)]_{ij} = d_i y_{ij} - y_{ij} d_j$$

and in general the expression $d_j y_{ij}$ is not defined, so we cannot imitate the expression $(i - j)y_{ij}$ from the basic example. The general strategy in the sections to come is to replace D by a perturbed operator $\overline{D}$ with pure point spectrum, such that $D - \overline{D}$ is bounded and satisfies $\|D - \overline{D}\| \leq 1/2$. We define $\overline{D}$ as a Borel function of D.

$$\overline{D} := \text{closure}\Big(\sum_{n\in\mathbb{Z}} ne_n\Big), \tag{4.1}$$

and then we may define a bounded operator b as the infinite strong operator sum below, such that

$$b := \sum_{n\in\mathbb{Z}} (De_n - ne_n), \quad \|b\| \leq \frac{1}{2}, \quad D = \overline{D} + b. \tag{4.2}$$

It is now easy to check that for any bounded operator y we have

$$[m(\overline{D}), m(y)]_{ij} = (i-j)y_{ij},$$

so differentiation with respect to $\overline{D}$ may be expressed as a Schur product.

5 On $\|[g(D), y]\|$ for Some Borel Functions $g(t)$

The usage of the Schur product, inside the space of matrices $\mathscr{M}_D$ in the study of commutators of the type $[g(D), x]$, is easily demonstrated when $g(t)$ is Hölder continuous, i.e. there exist positive constants α and C such that for any real numbers s and t we have $|g(s) - g(t)| \leq C|s-t|^\alpha$. In this case we say that $g(t)$ is (α, C) Hölder continuous. However, it turns out that the proof of such a result does not need as strong a property as Hölder continuity, instead we only need a property which we define below and denote Hölder boundedness.

Definition 5.1 Let $\alpha > 0$, $A \geq 0$, $B \geq 0$ and $g(t)$ be a complex Borel function on $\mathbb{R}$, then $g(t)$ is said to be (α, A, B) Hölder bounded if

$$\forall\, s, t \in \mathbb{R}: \quad |g(s) - g(t)| \leq A + B|s-t|^\alpha.$$

We have searched the literature for places where this property has been used or studied, but so far without any success.

It should be remarked, that the methods we propose also may fit to situations where $g(t)$ is only defined and Hölder bounded on a subset of $\mathbb{R}$ containing the spectrum of D, but the details needed for a general theorem seems to be somewhat complex, so we suggest that possible extensions in this directions are made on an *ad hoc* basis, if needed.

Theorem 5.2 *Let D be a self-adjoint operator on a Hilbert space and let $g(t)$ be an (α, A, B) Hölder bounded Borel function on $\mathbb{R}$. Let n be the smallest integer such that $n > \alpha + 1/2$ and y be a bounded operator which is n-times weakly D-differentiable. Then y is weakly $g(D)$−differentiable and*

$$\|[g(D), y]\| \leq 2(A+B)\left(\|y\| + \sqrt{\frac{n-\alpha}{2n-2\alpha-1}} \sum_{k=0}^{n} \binom{n}{k} \|\delta^k(y)\|\right).$$

Proof We will use the *discrete* approximant $\overline{D}$ as introduced in (4.1), and we note from (4.2) that the difference $\overline{D} - D$ is bounded, with closure b and of norm at most $1/2$. Then by the functional calculus for D we find that the difference $g(\overline{D}) - g(D)$ is densely defined and bounded of norm at most $A + B(1/2)^\alpha \leq A + B$. This implies,

that for any bounded operator y we have

$$\|[(m(g(D) - m(g(\overline{D}))), m(y)]\| \leq 2(A + B)\|y\|. \tag{5.1}$$

The Schur multiplier we need will be given as the matrix

$$S = (S_{ij})_{(i,j\in\mathbb{Z})}: \quad S_{ij} = \begin{cases} 0 & \text{if } i = j \\ \frac{g(i)-g(j)}{(i-j)^n} & \text{if } i \neq j \end{cases}, \tag{5.2}$$

and then for any (x_{ij}) in $\mathscr{M}_D$ we can express the matrix $[m(g(\overline{D})), x]$ as the Schur product $S * [m(\overline{D}), [m(\overline{D}), \dots, [m(\overline{D}), x] \dots]])$, with n commutators. In order to be able to use Corollary 2.4 we need to compute an estimate for the row norm of S and we get for any i in $\mathbb{Z}$ that

$$\begin{aligned} \sum_{j\in\mathbb{Z}} |S_{ij}|^2 &\leq 2\sum_{k=1}^{\infty}\left(\frac{A + Bk^{\alpha}}{k^n}\right)^2 \\ &\leq 2(A + B)^2\sum_{k=1}^{\infty} k^{2\alpha-2n} \\ &\leq 2(A + B)^2(1 + \frac{1}{2n - 2\alpha - 1}) \\ &= 4(A + B)^2\frac{n - \alpha}{2n - 2\alpha - 1}, \end{aligned} \tag{5.3}$$

so the row-norm is dominated as $\|S\|_r \leq 2(A + B)\sqrt{(n - \alpha)/(2n - 2\alpha - 1)}$.

This inequality may be used to compare expressions like $\|[g(\overline{D}), y]\|$ and $\|[\overline{D}, [\overline{D}, \dots, [\overline{D}, y] \dots]]\|$, but we are interested in the terms based on D instead of $\overline{D}$, so we will introduce the operators $\overline{d}$ and f on $\mathscr{M}_D$ which are defined by

$$\begin{aligned} \forall x \in \mathscr{M}_D: &\quad \overline{d}(x) := m(\overline{D})x - xm(\overline{D}) = [m(\overline{D}), x] \\ \forall x \in \mathscr{M}_D: &\quad f(x) := m(b)x - xm(b) = [m(b), x], \text{ and} \\ \forall x \in \mathscr{M}_D: &\quad x \text{ bounded } \implies f(x) \text{ bounded and } \|f(x)\| \leq \|x\| \\ \forall x \in \mathscr{M}_D: &\quad d(x) = \overline{d}(x) + f(x). \end{aligned} \tag{5.4}$$

The operators $d, f, \overline{d}$ on the linear space $\mathscr{M}_D$ all commute since both $\overline{D}$ and b are functions of D, and any left multiplication operator commutes with any right multiplication operator. The norm estimate above follows since $\|b\| \leq 1/2$. By elementary algebra we get that

$$\overline{d}^n = \sum_{k=0}^{n}\binom{n}{k}(-f)^{n-k}d^k$$

and if y is n-times weakly D-differentiable then it is also n-times weakly $\overline{D}$-differentiable and satisfies

$$\|\overline{d}^n(m(y))\| \leq \sum_{k=0}^{n} \binom{n}{k} \|\delta^k(y)\| \tag{5.5}$$

Now assume that n is the least integer such that $n > \alpha + 1/2$, and y is bounded and n-times weakly D-differentiable then, even if some terms are infinite, we may by (5.1) and (5.5) estimate as follows

$$\begin{aligned}
\|[m(g(D)), m(y)]\| &\leq \|[\big(m(g(D)) - m(g(\overline{D}))\big), m(y)]\| + \|[m(g(\overline{D})), m(y)]\| \\
&\leq \|[m(g(\overline{D})), m(y)]\| + 2(A+B)\|y\| \\
&= \|(S_{ij}) * \overline{d}^n(m(y))]\| + 2(A+B)\|y\| \\
&\leq 2(A+B)\sqrt{(n-\alpha)/(2n-2\alpha-1)}\|\overline{d}^n(m(y))\| + 2(A+B)\|y\| \\
&\leq 2(A+B)\Bigg(\|y\| + \sqrt{(n-\alpha)/(2n-2\alpha-1)}\sum_{k=0}^{n}\binom{n}{k}\|\delta^k(y)\|\Bigg).
\end{aligned} \tag{5.6}$$

Hence we see that $[m(g(D)), m(y)]$ is bounded in $\mathscr{M}_D$, so y is weakly $g(D)-$differentiable and the theorem follows. □

5.1 On $\|[g(D), y]\|$ for Some Absolutely Continuous Functions $g(t)$

In this subsection $g(t)$ is assumed to be a complex valued absolutely continuous function on $\mathbb{R}$ which has a certain type of bound on its growth. To be more precise we assume that $g(t)$ has a locally integrable derivative $g'(t)$ which may be written as a sum of an essentially bounded function $u(t)$ and an integrable function $\ell(t)$. There exists a theory of interpolation in Banach spaces, [8], and it is well known that it is possible to define a Banach space $L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ consisting of equivalence classes of sums $\ell(t) + u(t)$ with $\ell(t)$ integrable and $u(t)$ essentially bounded. The norm of an element $h(t)$ in the Banach space $L^1(\mathbb{R}) + L^\infty(\mathbb{R})$ is given by

$$\|h\|_{(L^1+L^\infty)} := \inf\{\|\ell\|_1 + \|u\|_\infty \, : \ell \in \mathscr{L}^1(\mathbb{R}), u \in \mathscr{L}^\infty(\mathbb{R}) \text{ and } h = \ell + u, \text{ a.e.}\}.$$

Before we formulate our theorem we will like to remind you that for any $p \geq 1$ we have $L^p \subseteq L^1 + L^\infty$ such that for f in $\mathscr{L}^p$ we have $\|f\|_{(L^1+L^\infty)} \leq \|f\|_p^p + 1$. This is an easy consequence of the fact that if the set U is defined as $U := \{t \in \mathbb{R} :$

$|f(t)| \leq 1\}$ then f may be written as $f * (1 - 1_U) + f * 1_U$ which is a decomposition with the desired properties.

Theorem 5.3 *Let D be a self-adjoint operator on a Hilbert space H and y a bounded operator on H. Let $g(t)$ be a complex absolutely continuous function on $\mathbb{R}$. If $g(t)$ has a derivative $g'(t)$ in $\mathscr{L}^1(\mathbb{R}) + \mathscr{L}^\infty(\mathbb{R})$ and y is in $C^2(D)$ then y is weakly $g(D)-$differentiable and*

$$\|[g(D), y]\| \leq \|g'\|_{(L^1+L^\infty)}\big(4\|y\| + 4\|\delta(y)\| + 2\|\delta^2(y)\|\big)$$

Proof We start by showing that $g(t)$ is Hölder bounded, so let $\varepsilon > 0$ and let $\ell(t)$ in $\mathscr{L}^1(\mathbb{R})$ and $u(t)$ in $\mathscr{L}^\infty(\mathbb{R})$ be such that $g'(t) = \ell(t) + u(t),$ a.e. and $\|\ell\|_1 + \|u\|_\infty \leq \|g'\|_{(L^1+L^\infty)} + \varepsilon$. Then for any s, t in $\mathbb{R}$ we have

$$\begin{aligned} |g(s) - g(t)| &= |\int_t^s g'(v)dv| \\ &\leq |\int_t^s \ell(v)dv| + |\int_t^s u(v)dv| \\ &\leq \|\ell\|_1 + (\|u\|_\infty)|s - t|, \end{aligned} \tag{5.7}$$

so $g(t)$ is Hölder bounded with the constants $(1, \|\ell\|_1, \|u\|_\infty)$ We can then apply Theorem 5.2 to obtain

$$\begin{aligned} \|[g(D), y]\| &\leq 2(\|\ell\|_1 + \|u\|_\infty)\big(2\|y\| + 2\|\delta(y)\| + \|\delta^2(y)\|\big) \\ &\leq (\|g'\|_{(L^1+L^\infty)} + \varepsilon)\big(4\|y\| + 4\|\delta(y)\| + 2\|\delta^2(y)\|\big), \end{aligned} \tag{5.8}$$

and the theorem follows. □

We may obtain a result which only depends on $\delta(y)$ if we assume that $g'(t)$ is in $\mathscr{L}^p$ for a p in the interval $[1, 2[$.

Proposition 5.4 *Let D be a self-adjoint operator on a Hilbert space H, y a bounded operator on H, $1 \leq p < 2$ and $g(t)$ an absolutely continuous function on $\mathbb{R}$ such that $g'(t)$ is in $L^p(\mathbb{R})$. If y is weakly D-differentiable, then y is weakly $g(D)$ differentiable and* $\|[g(D), y]\| \leq 2\|g'\|_p\Big(\big(1 + 1/\sqrt{2-p}\big)\|y\| + \big(1/\sqrt{2-p}\big)\|\delta(y\|\Big)$.

Proof The cases $p = 1$ and $1 < p < 2$ have different proofs, and we will first assume that $p = 1$. Then for any pair s, t of real numbers

$$|g(s) - g(t)| = |\int_t^s g'(v)dv| \leq \|g'\|_1$$

so $g(t)$ is Hölder bounded for any $\alpha > 0$ with constants $(\alpha, \|g'\|_1, 0)$. Then By Theorem 5.2 we have for $0 < \alpha < 1/2$ that

$$\|[g(D), y]\| \le 2\|g'\|_1\big(\|y\| + \sqrt{\frac{1-\alpha}{1-2\alpha}}(\|y\| + \|\delta(y)\|)\big),$$

and if we let α decrease to 0, the result follows in the case $p = 1$.

When $1 < p < 2$ Hölder's inequality gives

$$|g(t) - g(s)| = |\int_s^t g'(v)dv| \le \|g'\|_p|t-s|^{(p-1)/p},$$

so $g(t)$ is Hölder continuous with constants $((p-1)/p, \|g'\|_p)$. Hence Theorem 5.2 shows that we may use $n = 1$ to obtain that if y is weakly D-differentiable then it is weakly $g(D)$ differentiable with

$$\begin{aligned}\|[g(D), y]\| &\le 2\|g'\|_p\big(\|y\| + \frac{1}{\sqrt{2-p}}(\|y\| + \|\delta(y)\|)\big) \\ &= 2\|g'\|_p\Big(\big(1 + \frac{1}{\sqrt{2-p}}\big)\|y\| + \frac{1}{\sqrt{2-p}}\|\delta(y)\|\Big)\end{aligned} \tag{5.9}$$

and the proposition follows. □

5.2 *Relations Between Commutators* $[D, y]$ *and Commutators* $[\log(D), y]$

In this subsection we will show how Proposition 5.4 may be applied to the situation where we study $\log(D)$ for a positive, possibly unbounded operator D on a Hilbert space H such that D is either invertible or the point $\{0\}$ is an isolated point in the spectrum $\sigma(D)$. In either case there is a real number β which is the smallest positive value in $\sigma(D)$, and this value will play an important role in the estimates we are giving below.

For any $\beta > 0$ we then define an absolutely continuous real function $g_\beta(t)$ on $\mathbb{R}$ which will make the use of Proposition 5.4 possible.

We define

$$\forall \beta > 0: \qquad g_\beta(t) := \begin{cases} \log(t) & \text{if } t \ge \beta \\ \log(\beta) & \text{if } -\infty < t < \beta \end{cases},$$

and then we see that the conditions for $g_\beta(t)$ as given in Proposition 5.4 are satisfied for $p = 3/2$ and that

$$\|g'_\beta\|_{(3/2)} = 2^{(2/3)}\left(\frac{1}{\beta}\right)^{(1/3)}. \tag{5.10}$$

As an immediate consequence of Proposition 5.4 we get the following proposition

Proposition 5.5 *Let D be a positive self-adjoint operator on a Hilbert space H, such that there exists a smallest positive value β in the spectrum, and let y be a bounded operator on H. If y is weakly D-differentiable then it is weakly $g_\beta(D)$ differentiable and*

$$\|[g_\beta(D), y]\| \leq \left(\frac{1}{\beta}\right)^{(1/3)}\Big(8\|y\| + 5\|\delta(y)\|\Big).$$

We will also introduce the function $\widetilde{\log}(t)$ which is defined by

$$\widetilde{\log}(t) := \begin{cases} \log(t) \text{ if } t > 0 \\ 0 \text{ if } t \leq 0, \end{cases}$$

and we find that if D is positive and invertible then $g_\beta(D) = \widetilde{\log}(D)$. In order to investigate commutators with $\widetilde{\log}(D)$ when D is positive and not invertible, we let E_0 denote the spectral projection onto the kernel of D, and then we find that $D+\beta E_0$ is invertible plus

$$\widetilde{\log}(D) = g_\beta(D) - \log(\beta)E_0, \tag{5.11}$$

so based on this we see that a bounded operator y is weakly $\widetilde{\log}(D)-$differentiable if and only if it is weakly $g_\beta(D)-$differentiable, and then we can estimate the norm $\|[\widetilde{\log}(D), y]\|$, so we obtain

Theorem 5.6 *Let D be a positive operator on a Hilbert space such that there exist a minimal positive value β in $\sigma(D)$ and y a bounded operator on H. If y is weakly D-differentiable, then it is weakly $\widetilde{\log}(D)-$differentiable and*

(i) If D is invertible then

$$\|[\widetilde{\log}(D), y]\| \leq \left(\frac{1}{\beta}\right)^{(1/3)}\Big(8\|y\| + 5\|\delta(y)\|\Big).$$

(ii) If D is not invertible then

$$\|[\widetilde{\log}(D), y]\| \leq \big(8(1/\beta)^{(1/3)} + |\log(\beta)|\big)\|y\| + 5(1/\beta)^{(1/3)}\|\delta(y)\|.$$

Proof The case (i) follows directly from Proposition 5.5 since, when D is invertible, $g_\beta(D) = \widetilde{\log}(D)$. To settle (ii). let us recall, that for any bounded operator y we have

$$[E_0, y] = E_0 y(I - E_0) - (I - E_0) y E_0,$$

so $\|[E_0, y]\| \leq \|y\|$, hence a combination of the equation (5.11) and Proposition 5.5 implies the second estimate, so the theorem follows. □

It may be worth to notice that when D is invertible, then we get for any positive real s, that $\widetilde{\log}(sD) = \log(s)I + \widetilde{\log}(D)$, so for $s > 0$ we have

$$\|[\widetilde{\log}(D), y]\| = \|[\widetilde{\log}(sD), y]\| \quad \leq \left(\frac{1}{s\beta}\right)^{(1/3)} \left(8\|y\| + s(5\|\delta(y)\|)\right). \tag{5.12}$$

The minimum of the right hand side of (5.12) over $s > 0$ is $12(5/4)^{(1/3)}\beta^{-(1/3)}\|y\|^{(2/3)}\|\delta(y)\|)^{(1/3)}$, so we have proven the following result.

Corollary 5.7 *If D is invertible then*

$$\|[\widetilde{\log}(D), y]\| \leq 13\beta^{-(1/3)}\|y\|^{(2/3)}\|\delta(y)\|^{(1/3)}.$$

Since $g'_\beta(t)$ is in $L^p(\mathbb{R})$ for any $p > 1$ we could have chosen to optimize over $p > 1$ too, but we have decided to keep this article at a reasonable length.

5.3 *Norms of Commutators* $[|D|, y]$

Theorem 5.2 may be applied to the function $|t|$ which is $(1, 1)$ Hölder continuous and then it is $(1, 0, 1)$ Hölder bounded, so we get right away the following result

Proposition 5.8 *Let y be in $C^2(D)$. Then y is in $C^1(|D|)$ and*

$$\|\delta_{|D|}(y)\| \leq 4\|y\| + 4\|\delta_D(y)\| + 2\|\delta_D^2(y)\|.$$

This result shows by induction, that for any natural number n there must exist positive reals $A_0, \ldots, A_{2n}$ such that if y is in $C^{2n}(D)$ then y is in $C^n(|D|)$ and

$$\|\delta_{|D|}^n(y)\| \leq A_0\|y\| + A_1\|\delta_D(y)\| + \cdots + A_{2n}\|\delta_D^{2n}(y)\|.$$

However, refinements of the methods used in the proof of Theorem 5.2, show that we can do much better. We have obtained several inequalities, which are not easily directly comparable, by changing the grid length in the construction of $\overline{D}$. If the grid length is kept as 1, then a direct application of the methods from above shows that we can obtain

Theorem 5.9 *Let D be an unbounded operator on a Hilbert space H, y a bounded operator on H and n a natural number. If y is $n+1$ times weakly D-differentiable, then y is n times weakly $|D|-$differentiable and*

$$\|\delta^n_{|D|}(y)\| \leq 2^n \frac{\pi}{\sqrt{3}}\|y\| + \frac{\pi}{\sqrt{3}} \sum_{l=1}^{n+1} \binom{n+1}{l} 2^{(n+1-l)} \|\delta^l_D(y)\|.$$

Proof The arguments are similar to the ones used in the proof of Theorem 5.2, so we will leave out a few details. For a diagonal element z in $\mathscr{M}_D$ we will let d_z denote the linear mapping on $\mathscr{M}_D$ given by $d_z(x)_{ij} = z_{ii}x_{ij} - x_{ij}z_{jj}$. The operators z we will use, will all be of the form $z = m(f(D))$ for some Borel function $f(t)$, so they will all commute as operators on $\mathscr{M}_D$. We will recall the definitions of $\overline{D}$ and b from (4.1) and (4.2), but we will add the definition of c as the bounded operator which satisfies $|D| = |\overline{D}| + c$. Then c satisfies $\|c\| \leq 1/2$. For each natural number k we let $S(k) := (S(k)_{ij})$ denote the matrix in $\mathscr{S}$ which is given by

$$S(k)_{ij} = \begin{cases} 0 & \text{if } i = j \\ \frac{(|i|-|j|)^k}{(i-j)^{k+1}} & \text{if } i \neq j \end{cases} .$$

Since $||i| - |j|| \leq |i - j|$, the square of the row norm equals $2\sum_{\mathbb{N}} j^{-2} = \pi^2/3$. We will let F_k denote the operator on $\mathscr{M}_D$ which consists in Schur multiplication by $S(k)$. Then we have the following sequence of identities

$$\begin{aligned}
& d^n_{m(|D|)} = (d_{m(|\overline{D}|)} + d_{m(c)})^n \\
=& d^n_{m(c)} + \sum_{k=1}^{n} \binom{n}{k} d^{n-k}_{m(c)} d^k_{m(|\overline{D}|)} \\
=& d^n_{m(c)} + \sum_{k=1}^{n} \binom{n}{k} d^{n-k}_{m(c)} F_k d^{k+1}_{m(\overline{D})} \\
=& d^n_{m(c)} + \sum_{k=1}^{n} \binom{n}{k} d^{n-k}_{m(c)} F_k \big(d_{m(D)} - d_{m(b)}\big)^{k+1} \\
=& d^n_{m(c)} + \sum_{k=1}^{n} \binom{n}{k} d^{n-k}_{m(c)} F_k (-d_{m(b)})^{k+1} \\
&+ \sum_{k=1}^{n} \sum_{l=1}^{k+1} \binom{n}{k} \binom{k+1}{l} d^{n-k}_{m(c)} F_k (-d_{m(b)})^{(k+1-l)} d^l_{m(D)}
\end{aligned} \tag{5.13}$$

Since for a bounded operator y we have $\|cy - yc\| \leq \|y\|$, $\|by - yb\| \leq \|y\|$ and $\|F_k(m(y))\| \leq \pi/\sqrt{3}\|y\|$ we may start computing. So let y be $n+1$ times weakly D-differentiable. Then the identities above show

$$\begin{aligned}
\|d^n_{m(|D|)}(y)\| &\leq 2^n\pi/\sqrt{3}\|y\| + \pi/\sqrt{3}\sum_{k=1}^{n}\sum_{l=1}^{k+1}\binom{n}{k}\binom{k+1}{l}\|\delta^l(y)\| \\
&= 2^n\pi/\sqrt{3}\|y\| + \pi/\sqrt{3}\sum_{l=1}^{n+1}\sum_{k=l-1}^{n}\binom{n}{k}\binom{k+1}{l}\|\delta^l(y)\| \\
&\leq 2^n\pi/\sqrt{3}\|y\| + \pi/\sqrt{3}\sum_{l=1}^{n+1}\sum_{u=0}^{n+1-l}\binom{n+1}{l}\binom{n+1-l}{u}\|\delta^l(y)\| \\
&= 2^n\pi/\sqrt{3}\|y\| + \pi/\sqrt{3}\sum_{l=1}^{n+1}\binom{n+1}{l}2^{(n+1-l)}\|\delta^l(y)\|
\end{aligned} \tag{5.14}$$

and the theorem follows. □

If the grid length is made smaller, say of size 1/2, then the powers of 2 will disappear from the theorem above, but the row norm of $S(k)$ will double. We have left these computations out, partly because they are rather long, and partly because we do not know if there exists an optimal grid length.

References

1. W. O. Amrein, A. Boutet de Monvel and V. Georgescu. *C_0-Groups, commutator methods and spectral theory of N-body Hamiltonians.* Birkhäuser, 1996.
2. G. Bennett. *Schur multipliers.* Duke Math. J. 44, 1977, 603–629.
3. E. Christensen. *On weakly D-differentiable operators.* To appear in Expo. Math., http://dx.doi.org/10.1016/j.exmath.2015.03.002
4. E. Christensen. *Higher weak derivatives and reflexive algebras of operators.* To appear in Contemp. Math. volume dedicated to R. V. Kadison, 2015. http://arxiv.org/pdf/1504.03521.pdf
5. A. Connes. *Noncommutative geometry.* Academic Press, 1994.
6. E. G. Effros and Z.-J.- Ruan. *Operator spaces.* London Math. Soc. monographs 23, Oxford U. P., 2000.
7. R. V. Kadison and J. R. Ringrose. *Fundamentals of the theory of operator algebras.* Academic Press, 1983.
8. J. Lindenstrauss and L. Tzafriri. *Classical Banach spaces.* Ergebnisse Math. 97, Springer 1979.
9. J. van Neerven. *The adjoint of a semigroup of linear operators.* Springer, Lect. Notes Math. 1529, 1992.
10. J. von Neumann. *Mathematische Grundlagen der Quantenmechanik.* Springer Verlag, 1932, 1968, 1996.

11. V. Paulsen. *Completely bounded maps and operator algebras.* Cambridge University Press, 2002.
12. G. K. Pedersen. *A commutator inequality.* In *Operator algebras, mathematical physics and low dimensional topology,* 233–235. ed. R. Herman and B. Tanbay. Res. Notes Math. 5., A. K. Peters USA, 1993.
13. R. S. Phillips. *The adjoint semi-group.* Pacific J. Math. 5, 1955, 269–283.
14. G. Pisier. *Similarity problems and completely bounded maps.* Springer Lect. Notes Math. 1618, ed. 2001.
15. G. Pisier. *Introduction to operator space theory.* London Mathematical Society Lecture Note Series. 294, 2003.

C*-Algebras Associated with Algebraic Actions

Joachim Cuntz

Abstract This is a survey of work in which the author was involved in recent years. We consider C*-algebras constructed from representations of one or several algebraic endomorphisms of a compact abelian group—or, dually, of a discrete abelian group. In our survey we do not try to describe the entire scope of the methods and results obtained in the original papers, but we concentrate on the important thread coming from the action of the multiplicative semigroup of the ring of integers in an algebraic number field, or more generally of a Dedekind ring, on its additive group. Representations of such actions give rise to particularly intriguing problems and the study of the corresponding C*-algebras has motivated many of the new methods and general results obtained in this area.

1 Introduction

By an algebraic action we mean here an action of a semigroup by algebraic endomorphisms of a compact abelian group or, dually, by endomorphisms of a discrete abelian group. Such actions are much studied in ergodic theory but they also give rise to interesting C*-algebras. In fact, quite a few of the standard examples of simple C*-algebras such as $\mathcal{O}_n$-algebras, Bunce-Deddens algebras, UHF-algebras etc. arise from canonical representations of such endomorphisms. But the class of C*-algebras obtained from general algebraic actions is much vaster and exhibits new interesting phenomena.

We start our survey with the discussion, following [12], of the C*-algebra $\mathfrak{A}[\alpha]$ generated by the so called Koopman representation on L^2H of a single endomorphism α, satisfying natural conditions, of a compact abelian group H, together with the natural representation of $C(H)$. This C*-algebra is always simple purely infinite and can be described by a natural set of generators and relations. It contains a canonical maximal commutative C*-algebra $\mathcal{D}$ with spectrum a Cantor space. This subalgebra is generated by the range projections $s^n s^{n*}$, where s is the isometry implementing the given endomorphism, and by their conjugates $u_\gamma s^n s^{n*} u_\gamma^*$,

J. Cuntz (✉)
Mathematisches Institut, Münster University, Münster, Germany
e-mail: cuntz@uni-muenster.de

T.M. Carlsen et al. (eds.), *Operator Algebras and Applications*, Abel Symposia 12,
DOI 10.1007/978-3-319-39286-8_6

under the unitaries u_γ given by the characters γ of H. Then, the subalgebra $\mathscr{B}$ generated by $\mathscr{D}$ together with the u_γ is of Bunce-Deddens type and simple with unique trace. Moreover, $\mathfrak{A}[\alpha]$ can be considered as a crossed product of $\mathscr{B}$ by a single endomorphism.

The next case we consider is the C*-algebra generated analogously by the Koopman representation of a family of commuting endomorphisms. We consider the important special case of endomorphisms arising from the ring of integers R in a number field K. The multiplicative semigroup $R^\times$ acts by commuting endomorphisms on the additive group $R \cong \mathbb{Z}^n$ or equivalently on the dual group $\widehat{R} \cong \mathbb{T}^n$ (n being the degree of the field extension K over $\mathbb{Q}$). The commutative semigroup $R^\times$ has a non-trivial structure and acts by interesting endomorphisms on $\mathbb{T}^n$. The study of the C*-algebra $\mathfrak{A}[R]$ generated by the Koopman representation in this situation goes back to [5] and was originally motivated by connections to Bost-Connes systems [2].

Again, $\mathfrak{A}[R]$ is simple purely infinite and is described by natural generators and relations. It has analogous subalgebras $\mathscr{D}$ and $\mathscr{B}$, and $\mathfrak{A}[R]$ can be viewed as a semigroup crossed product $\mathscr{B} \rtimes R^\times$. The new and challenging problem is the computation of the K-theory of $\mathfrak{A}[R]$. The key to this computation is a duality result for adele-groups and corresponding crossed products, [10].

Since $\mathfrak{A}[R]$ is generated by the Koopman representation, on $\ell^2 R$, of the semidirect product semigroup $R \rtimes R^\times$, the next very natural step in our program is the consideration of the C*-algebra generated by the natural representation of this semigroup on $\ell^2(R \rtimes R^\times)$ rather than on $\ell^2 R$, i.e. of the left regular C*-algebra $C^*_\lambda(R \rtimes R^\times)$. This algebra is still purely infinite but no longer simple. It can be described by natural generators and relations. The algebra $\mathfrak{A}[R]$ is a quotient of $C^*_\lambda(R \rtimes R^\times)$ and the latter algebra is defined by relaxing the relations defining $\mathfrak{A}[R]$ in a systematic way. The best way to do so is to add a family of projections, indexed by the ideals of the ring R, as additional generators and to incorporate those into the relations. This way of defining the relations also guided Xin Li in his description of the left regular C*-algebras for more general semigroups [19].

The (non-trivial) problem of computing the K-theory of $C^*_\lambda(R \rtimes R^\times)$ turned out to be particularly fruitful [7, 8]. It led to a powerful new method for computing the K-groups, for regular C*-algebras of more general semigroups and of crossed products by automorphic actions of such more general semigroups, as well as for crossed products of certain actions of groups on totally disconnected spaces. In the special case of $C^*_\lambda(R \rtimes R^\times)$ we get the interesting result that the K-theory is described by a formula that involves the basic number theoretic structure of the number field K, namely the ideal class group and the action of the group of units (invertible elements in R) on the additive group of an ideal.

Finally, we include a brief discussion of the rich KMS-structure on $C^*_\lambda(R \rtimes R^\times)$ for the natural one-parameter action on this C*-algebra. Just as the K-theory for $C^*_\lambda(R \rtimes R^\times)$, this structure is related to the number theoretic invariants of R, resp. K.

Our goal in this survey is limited. We try to describe a leitmotif in this line of research and to explain the connections and similarities between the various results. The original articles contain much more information and many additional finer,

more sophisticated and more general results which we omit completely. We also do not describe the results in the order they were obtained originally, but rather in the order which seems more systematic with hindsight.

2 Single Algebraic Endomorphisms

Let H be a compact abelian group and $G = \widehat{H}$ its dual discrete group. We assume that G is countable. Let α be a surjective endomorphism of H with finite kernel. We denote by φ the dual endomorphism $\chi \mapsto \chi \circ \alpha$ of G (i.e. $\varphi = \widehat{\alpha}$). By duality, φ is injective and has finite cokernel, i.e. the quotient $G/\varphi G$ will be finite. Both α and φ induce isometric endomorphisms s_α and s_φ of the Hilbert spaces L^2H and ℓ^2G, respectively. This isometric representation of α on L^2H is called the Koopman representation in ergodic theory.

We will also assume that

$$\bigcap_{n\in\mathbb{N}} \varphi^n G = \{0\}$$

which, by duality, means that

$$\bigcup_{n\in\mathbb{N}} \operatorname{Ker} \alpha^n$$

is dense in H (this implies in particular that H and G cannot be finite). These conditions on α are quite natural and for instance apply to the usual examples considered in ergodic theory. We list a few important examples of compact groups and endomorphisms satisfying our conditions at the end of this section.

We want to describe the C*-algebra $C^*(s_\alpha, C(H))$ generated in $\mathscr{L}(L^2H)$ by $C(H)$, acting by multiplication operators, and by the isometry s_α. Via Fourier transform it is isomorphic to the C*-algebra $C^*(s_\varphi, C^*G)$ generated in $\mathscr{L}(\ell^2G)$ by C^*G, acting via the left regular representation, and by the isometry s_φ. These two unitarily equivalent representations are useful for different purposes.

Now, $C^*(s_\varphi, C^*G)$ is generated by the isometry $s = s_\varphi$ together with the unitary operators u_g, $g \in G$ and these operators satisfy the relations

$$u_g u_h = u_{g+h} \qquad s u_g = u_{\varphi(g)} s \qquad \sum_{g\in G/\varphi G} u_g s s^* u_g^* = 1 \tag{2.1}$$

Definition 2.1 Let H, G and α, φ be as above. We denote by $\mathfrak{A}[\varphi]$ the universal C*-algebra generated by an isometry s and unitary operators u_g, $g \in G$ satisfying the relations (2.1).

It is shown in [12] that $\mathfrak{A}[\varphi] \cong C^*(s_\alpha, C(H)) \cong C^*(s_\varphi, C^*G)$, i.e. that the natural map from the universal C*-algebra to the C*-algebra generated by the concrete Koopman representation is an isomorphism. Particular situations of interest arise when $H = (\mathbb{Z}/n)^\infty$ with α the left shift (this gives rise to $\mathfrak{A}[\varphi] \cong \mathcal{O}_n$) or when $H = \mathbb{T}^n$.

Lemma 2.2 *The C*-subalgebra $\mathcal{D}$ of $\mathfrak{A}[\varphi]$ generated by all projections of the form $u_g s^n s^{*n} u_g^*$, $g \in G, n \in \mathbb{N}$ is commutative. Its spectrum is the "φ-adic completion"*

$$G_\varphi = \varprojlim_n G/\varphi^n G$$

It is an inverse limit of the finite spaces $G/\varphi^n G$ and becomes a Cantor space with the natural topology.

G acts on $\mathcal{D}$ via $d \mapsto u_g d u_g^*$, $g \in G$, $d \in \mathcal{D}$. This action corresponds to the natural action of the dense subgroup G on its completion G_φ via translation. The map $\mathcal{D} \to \mathcal{D}$ given by $x \mapsto sxs^*$ corresponds to the map induced by φ on G_φ.

From now on we will denote the compact abelian group G_φ by M. By construction, G is a dense subgroup of M. The dual group of M is the discrete abelian group

$$L = \varinjlim_n \mathrm{Ker}\,(\alpha^n : H \to H)$$

Because of the condition that we impose on α, L can be considered as a dense subgroup of H.

The groups M and L play an important role in the analysis of $\mathfrak{A}[\varphi]$. They are in a sense complementary to H and G. By Lemma 2.2, the C*-algebra $\mathcal{D}$ is isomorphic to $C(M)$ and to $C^*(L)$.

Theorem 2.3 *The C*-subalgebra B_φ of $\mathfrak{A}[\varphi]$ generated by $C(H)$ together with $C(M)$ (or equivalently by C^*G together with C^*L) is isomorphic to the crossed product $C(M) \rtimes G$. It is simple and has a unique trace.*

The map $x \mapsto sxs^*$ defines a natural endomorphism γ_φ of B_φ.

Theorem 2.4 *The algebra $\mathfrak{A}[\varphi]$ is simple, nuclear and purely infinite. Moreover, it is isomorphic to the semigroup crossed product $B_\varphi \rtimes_{\gamma_\varphi} \mathbb{N}$ (i.e. to the universal unital C*-algebra generated by B_φ together with an isometry t such that $txt^* = \gamma_\varphi(x)$, $x \in B_\varphi$).*

The fact that $\mathfrak{A}[\varphi]$ is a crossed product $B_\varphi \rtimes \mathbb{N}$ can be used to prove the following

Theorem 2.5 (cf. [12]) *The K-groups of $\mathfrak{A}[\varphi]$ fit into an exact sequence as follows*

$$\begin{tikzcd} K_*C(H) \arrow[r, "1-b(\varphi)"] & K_*C(H) \arrow[r] & K_*\mathfrak{A}[\varphi] \arrow[ll, bend left] \end{tikzcd} \tag{2.2}$$

where the map $b(\varphi) : K_*C(H) \to K_*C(H)$ satisfies $b(\varphi)\alpha_* = N(\alpha)\mathrm{id}$ with $N(\alpha) := |\mathrm{Ker}\,\alpha|$.

In [12], the analysis of $\mathfrak{A}[\varphi]$ and the formula (2.2) for its K-theory was also extended to the case where α is replaced by a so called rational polymorphism.

There are quite a few papers in the literature containing special cases or parts of the results described in this section. We mention only [15] where it was shown that $\mathfrak{A}[\varphi]$ is simple and characterized by generators and relations and [14] where in particular a formula similar to (2.2) was derived for an expansive endomorphism of $\mathbb{T}^n$—both papers using methods different from [12].

2.1 Examples

Here are some examples of endomorphisms in the class we consider.

1. Let $H = \prod_{k\in\mathbb{N}} \mathbb{Z}/n$, $G = \bigoplus_{k\in\mathbb{N}} \mathbb{Z}/n$ and α the one-sided shift on H defined by $\alpha((a_k)) = (a_{k+1})$.

   We obtain $M = \prod_{k\in\mathbb{N}} \mathbb{Z}/n \cong H$ and $L = \bigoplus_{k\in\mathbb{N}} \mathbb{Z}/n \cong G$. The algebra B_φ is a UHF-algebra of type n^∞ and $\mathfrak{A}[\varphi]$ is isomorphic to $\mathcal{O}_n$. It is interesting to note that the UHF-algebra B_φ is generated by two maximal abelian subalgebras both isomorphic to $C(M)$.
2. Let $H = \mathbb{T}$, $G = \mathbb{Z}$ and α the endomorphism of H defined by $\alpha(z) = z^n$. The algebra B_φ is a Bunce-Deddens-algebra of type n^∞ and $\mathfrak{A}[\varphi]$ is isomorphic to a natural subalgebra of the algebra $\mathcal{Q}_\mathbb{N}$ considered in [5]. In this case, we also get for B_φ the interesting isomorphism $C(\mathbb{Z}_n) \rtimes \mathbb{Z} \cong C(\mathbb{T}) \rtimes L$ where $\mathbb{Z}$ acts on the Cantor space $\mathbb{Z}_n$ by the odometer action (addition of 1) and L denotes the subgroup of $\mathbb{T}$ given by all n^k-th roots of unity, acting on $\mathbb{T}$ by translation.
3. Let $H = \mathbb{T}^n$, $G = \mathbb{Z}^n$ and α an endomorphism of H determined by an integral matrix T with non-zero determinant. We assume that the condition

$$\bigcap_{n\in\mathbb{N}} \varphi^n G = \{0\}$$

   is satisfied (this is in fact not very restrictive).

   The algebra B_φ is a higher-dimensional analogue of a Bunce-Deddens-algebra. In the case where H is the additive group of the ring R of algebraic integers in a number field of degree n and the matrix T corresponds to an element of R, the algebra $\mathfrak{A}[\varphi]$ is isomorphic to a natural subalgebra of the algebra $\mathfrak{A}[R]$ considered in the following section. It is also isomorphic to the algebra studied in [14].
4. As another natural example related to number theory consider the additive group of the polynomial ring $\mathbb{F}_p[t]$ over a finite field. An endomorphism satisfying our conditions is given by multiplication by a non-zero element in $\mathbb{F}_p[t]$. In this case $\mathfrak{A}[\varphi]$ is related to certain graph C*-algebras, see [11].
5. Let p and q be natural numbers that are relatively prime and γ the endomorphism of $\mathbb{T}$ defined by $z \mapsto z^p$.

We take

$$H = \lim_{\overleftarrow{\gamma}} \mathbb{T} \qquad G = \mathbb{Z}[\frac{1}{p}]$$

α_q the endomorphism of H induced by $z \mapsto z^q$ and φ_q the endomorphism of G defined by $\varphi_q(x) = qx$. These endomorphisms satisfy our hypotheses. We find that $M = \mathbb{Z}_q$ (the q-adic completion of $\mathbb{Z}$).

In all these examples one can work out the K-theory of $\mathfrak{A}[\varphi]$ using formula (2.2), see [12].

3 Actions by a Family of Endomorphisms, Ring C*-Algebras

It is a natural problem to extend the results of Sect. 2 to actions of a family (semigroup) of several commuting endomorphisms of a compact abelian group, satisfying the conditions of Sect. 2. It turns out that the structural results such as simplicity, pure infiniteness, canonical subalgebras carry over without problem. However the computation of the K-groups needs completely new ideas.

The most prominent example for us arises as follows. Let K be a number field, i.e. a finite algebraic extension of $\mathbb{Q}$. The ring of algebraic integers $R \subset K$ is defined as the integral closure of $\mathbb{Z}$ in K, i.e. as the set of elements $a \in K$ that annihilate some monic polynomial with coefficients in $\mathbb{Z}$. This ring is always a Dedekind domain (a Dedekind domain is by definition an integral domain in which every nonzero proper ideal factors into a product of prime ideals). It has many properties similar to the ordinary ring of integers $\mathbb{Z} \subset \mathbb{Q}$, but it is not a principal ideal domain in general. Its additive group is always isomorphic to $\mathbb{Z}^n$ where n is the degree of the field extension.

Consider the multiplicative semigroup $R^\times = R \setminus \{0\}$ of R. It acts as endomorphisms on the additive group R and thus also on the compact abelian dual group $\widehat{R} \cong \mathbb{T}^n$. Such endomorphisms of $\mathbb{T}^n$ are a frequent object of study in ergodic theory. If R is not a principal ideal domain, the semigroup $R^\times$ has an interesting structure.

As in Sect. 2 we consider the Koopman representation of $R^\times$ on $L^2\widehat{R} \cong \ell^2 R$.

Definition 3.1 We define the 'ring C*-algebra' $\mathfrak{A}[R]$ as the C*-algebra generated by $C(\widehat{R})$ and $R^\times$ on $L^2(\widehat{R})$ (or equivalently as the C*-algebra generated by the action of $C^*(R)$ and of $R^\times$ on $\ell^2 R$).

$\mathfrak{A}[R]$ is generated by the isometries s_n, $n \in R^\times$ and the unitaries $u_j, j \in R$. The s_n define a representation of the abelian semigroup $R^\times$ by isometries, the u_j define a representation of the abelian group R by unitaries and together they satisfy the relations

$$s_n u_k = u_{kn} s_n, \ k \in R, \ n, m \in R^\times \qquad \sum_{j \in R/nR} u_j s_n s_n^* u_{-j} = 1 \tag{3.1}$$

The basic analysis of the structure of $\mathfrak{A}[R]$ is completely parallel to the discussion in Sect. 2 (in fact historically the article [9] preceded [12]). One obtains

Theorem 3.2 (cf. [9]) *The C*-algebra $\mathfrak{A}[R]$ is simple purely infinite and nuclear. It is the universal C*-algebra generated by a unitary representation u of R together with an isometric representation s of $R^\times$ satisfying the relations* (3.1)*.*

As for $\mathfrak{A}[\varphi]$ in Sect. 2 there are canonical subalgebras $\mathscr{D}$ and $\mathscr{B}$ of $\mathfrak{A}[R]$. The spectrum of the commutative C*-algebra $\mathscr{D}$ is a Cantor space canonically homeomorphic to the maximal compact subring of the space of finite adeles for the number field K. The subalgebra $\mathscr{B}$ is generated by $\mathscr{D}$ together with the u_j, $j \in R$. It is simple and has a unique trace (a higher dimensional Bunce-Deddens type algebra). The general structure of C*-algebras associated like this with a ring has been developed further by Xin Li in [18].

In order to compute the K-groups for $\mathfrak{A}[R]$ the natural strategy would appear to be an iteration of the formula (2.2) of Theorem 2.5. Since the proof of formula (2.2) is based on the usual Pimsner-Voiculescu sequence this would amount to iterating this sequence in order to compute the K-groups for the crossed product by $\mathbb{Z}^n$ by a commuting family of n automorphisms. However this strategy immediately runs into problems since, assuming the K-groups for the crossed product by the first automorphism are determined, it is not at all clear how the second automorphism will act on these groups. In other words, there is a spectral sequence abutting to the K-theory for the crossed product by $\mathbb{Z}^n$, but it is useless for actual computations without further knowledge of the higher boundary maps in the spectral sequence. An analysis of relevant properties of the spectral sequence for actions as here is contained in [1].

The key to the computation of the K-groups for $\mathfrak{A}[R]$ in [10] is the following duality result.

Theorem 3.3 *Let $\mathbb{A}_f$ and $\mathbb{A}_\infty$ denote the locally compact spaces of finite, resp. infinite adeles of K both with the natural action of the additive group K. Then the crossed product C*-algebras $C_0(\mathbb{A}_f) \rtimes K$ and $C_0(\mathbb{A}_\infty) \rtimes K$ are Morita equivalent, equivariantly for the action of $K^\times$ on both algebras (with the inverted natural action on the second algebra, i.e. $K^\times$ acts on $\mathbb{A}_\infty$ not by multiplication but by division).*

Note that the space $\mathbb{A}_\infty$ is simply $\mathbb{R}^n$ where n is the degree of the field extension. From this theorem the K-groups of $\mathfrak{A}[R]$ can be computed, at least in the case where the only roots of unit in K are ± 1.

We explain this here only for the case where $K = \mathbb{Q}$, $R = \mathbb{Z}$. In this case everything becomes rather concrete. The spectrum of the canonical commutative subalgebra $\mathscr{D}$ is the profinite completion $\overline{\mathbb{Z}}$ of $\mathbb{Z}$ (we use here $\overline{\mathbb{Z}}$ rather than the more standard notation $\widehat{\mathbb{Z}}$ in order not to create confusion with the dual group of $\mathbb{Z}$). It is homeomorphic to the infinite product of the p-adic completions $\mathbb{Z}_p$ for all primes p in $\mathbb{Z}$. Moreover $\mathbb{A}_f$ is the restricted infinite product of the $\mathbb{Q}_p$ and $\mathbb{A}_\infty$ simply is $\mathbb{R}$.

Thus Theorem 3.3 gives a Morita equivalence between $C_0(\mathbb{A}_f)\rtimes\mathbb{Q}$ and $C_0(\mathbb{R})\rtimes\mathbb{Q}$. Moreover the first crossed product is Morita equivalent to the full corner $\mathscr{B} \cong C(\overline{\mathbb{Z}}) \rtimes \mathbb{Z}$.

Denote by $\mathscr{B}'$ the C*-algebra generated by $\mathscr{B}$ together with the symmetry s_{-1}, i.e. $\mathscr{B}' \cong \mathscr{B} \rtimes \mathbb{Z}/2$ for the action of s_{-1}. Since $\mathscr{B}' \cong (C(\overline{\mathbb{Z}}) \rtimes \mathbb{Z}) \rtimes \mathbb{Z}/2$ is an inductive limit of $C((\mathbb{Z}/n\mathbb{Z}) \rtimes \mathbb{Z}) \rtimes \mathbb{Z}/2$ and this latter algebra is isomorphic to $M_n(C^*(\mathbb{Z}\rtimes\mathbb{Z}/2))$ it is not difficult to compute the K-theory of $\mathscr{B}$ as $K_0(\mathscr{B}') = \mathbb{Z}\oplus\mathbb{Q}$ and $K_1(\mathscr{B}') = 0$.

Now, we can use the Pimsner-Voiculescu sequence to compute the K-theory of the crossed product $\mathfrak{A}_1 = \mathscr{B}' \rtimes \mathbb{N} = C^*(\mathscr{B}', s_2)$ as

$$K_0(\mathfrak{A}_1) = \mathbb{Z} \qquad K_1(\mathfrak{A}_1) = \mathbb{Z}$$

By a slight refinement of the statement in Theorem 3.3, $\mathfrak{A}_1$ is Morita equivalent to $(C_0(\mathbb{R}) \rtimes \mathbb{Q}) \rtimes (\mathbb{Z}/2 \times \mathbb{Z})$ where $\mathbb{Z}/2 \times \mathbb{Z}$ acts by multiplication by -1 and by 2.

Denote now by $\mathfrak{A}_n$ the C*-algebra generated by $\mathscr{B}'$ together with the $s_{p_1}, \ldots, s_{p_n}$, where $p_1, \ldots, p_n$ denote the first n prime numbers (with $p_1 = 2$). Then again $\mathfrak{A}_n$ is Morita equivalent to $(C_0(\mathbb{R}) \rtimes \mathbb{Q}) \rtimes (\mathbb{Z}/2 \times \mathbb{Z}^n)$, where $\mathbb{Z}/2$ acts by multiplication by -1 and $\mathbb{Z}^n$ by multiplication by $p_1, \ldots, p_n$. Moreover $\mathfrak{A}[R]$ is the inductive limit of the $\mathfrak{A}_n$.

We can now consider the canonical inclusions

$$\iota_n : C_0(\mathbb{R}) \rtimes (\mathbb{Z}/2 \times \mathbb{Z}^n) \to (C_0(\mathbb{R}) \rtimes \mathbb{Q}) \rtimes (\mathbb{Z}/2 \times \mathbb{Z}^n) \sim_{\text{Morita}} \mathfrak{A}_n \tag{3.2}$$

into the crossed product where we leave out the action of the additive $\mathbb{Q}$ by translation on the left hand side.

By the discussion above, ι_1 induces an isomorphism in K-theory. Now we obtain ι_{n+1} from ι_n by taking the crossed product by $\mathbb{Z}$ (acting by multiplication by p_{n+1}) on both sides in (3.2). Therefore, applying the Pimsner-Voiculescu sequence on both sides, we deduce, using the five-lemma, from the fact that ι_n induces an isomorphism on K-theory that the same holds for ι_{n+1}. The important point is that the action of $\mathbb{Z}^n$ on the left hand side is homotopic to the trivial action, simply because multiplication by $p_1, \ldots, p_n$ is homotopic to multiplication by 1 on $\mathbb{R}$. Therefore $K_*(\mathfrak{A}_n) \cong K_*((C_0(\mathbb{R}) \rtimes \mathbb{Z}/2) \otimes C^*\mathbb{Z}^n)$.

As a consequence we obtain

Theorem 3.4 ([10]) *The map ι_n induces an isomorphism on K-theory for all n. The K-theory of $\mathfrak{A}[R]$ is isomorphic to the K-theory of $(C_0(\mathbb{R}) \rtimes \mathbb{Z}/2) \rtimes \mathbb{Q}^\times$.*

Note that the K-theory of $C_0(\mathbb{R}) \rtimes \mathbb{Z}/2$ is the same as the one of $\mathbb{C}$ and that therefore the K-theory of $\mathfrak{A}[R]$ is the same as the one of an infinite-dimensional torus.

The argument that we sketched for $K = \mathbb{Q}$ works in a very similar, though somewhat more involved way for a number field with ± 1 as only roots of unit. In this case one has to determine the K-theory of $C_0(\mathbb{R}^n) \rtimes \mathbb{Z}/2$ rather than that of $C_0(\mathbb{R}) \rtimes \mathbb{Z}/2$. The case of an arbitrary number field K can be treated in the same fashion. The important difference comes from the more general group $\mu(K)$ of roots

of unit. For the computation one needs non-trivial information on the K-theory of the crossed product $C_0(\mathbb{R}^n) \rtimes \mu(K)$ and thus on the equivariant K-theory of $\mathbb{R}^n$ with respect to the action of $\mu(K)$. This non-trivial computation has been carried through by Li and Lück in [21] using previous work by Langer and Lück [17].

The analysis of the structure and of the K-theory of $\mathfrak{A}[R]$ can also be carried out in the case where R is a polynomial ring over a finite field (ring of integers in a certain function field). The structure of the C*-algebra in this case is more closely related to the example of the shift endomorphism of $(\mathbb{Z}/p\mathbb{Z})^\infty$ mentioned above and to certain Cuntz-Krieger algebras. Nevertheless for the computation of the K-theory one can again use the duality result in Theorem 3.3 and the result for the K-theory is again similar, [11].

4 Regular C*-Algebra for 'ax+b'-Semigroups

By definition, the ring C*-algebra $\mathfrak{A}[R]$ discussed in Sect. 3 is obtained from the natural representations of $C^*(R) \cong C(\widehat{R})$ and of the semigroup $R^\times$ on the Hilbert space $\ell^2 R \cong L^2(\widehat{R})$. Another way to view this is to say that it is defined by the natural representation of the semidirect product semigroup $R \rtimes R^\times$ on $\ell^2 R$.

Now, this semidirect product semigroup has an even more natural representation, given by the left regular representation on the Hilbert space $\ell^2(R \rtimes R^\times)$. The study of the left regular C*-algebra $C^*_\lambda(R \rtimes R^\times)$ was begun in [6]. This C*-algebra is no longer simple but still purely infinite and has an intriguing structure. In particular, it has a very interesting *KMS*-structure and the determination of its K-theory leads to new challenging problems.

The first obvious observation concerning $C^*_\lambda(R \rtimes R^\times)$ is that, just as $\mathfrak{A}[R]$, it is generated by a unitary representation $u_x, x \in R$ of the additive group R and a representation by isometries $s_a, a \in R^\times$ of the multiplicative semigroup $R^\times$ satisfying the additional relation $s_a u_x = u_{ax} s_a$. However the last relation $\sum_{x\in R/aR} u_x s_a s_a^* u_{-x} = 1$ in (3.1) becomes

$$\sum_{x\in R/aR} u_x s_a s_a^* u_{-x} \leq 1 \tag{4.1}$$

In fact, it turns out that this weakened relation (4.1) (of course together with the relations on the u_x, s_a in the previous paragraph) determines $C^*_\lambda(R \rtimes R^\times)$ in the case where R is a principal ideal domain. The general case however is more intricate. In general, it is still possible to describe $C^*_\lambda(R \rtimes R^\times)$ by natural defining relations. However the most natural way to do so uses an incorporation of the natural idempotents obtained as range projections of the partial isometries given by products of the u_x, s_a and their adjoints. It turns out that these range projections correspond exactly to the ideals of R.

The generators singled out in [6] then are $u_x, x \in R$; s_a, $a \in R^\times$; e_I, I a non-zero ideal in R. The relations are

1. The u_x are unitary and satisfy $u_x u_y = u_{x+y}$, the s_a are isometries and satisfy $s_a s_b = s_{ab}$. Moreover $s_a u_x = u_{ax} s_a$ for all $x \in R$, $a \in R^\times$.
2. The e_I are projections and satisfy $e_{I \cap J} = e_I e_J$, $e_R = 1$.
3. We have $s_a e_I = e_{aI} s_a$.
4. For $x \in I$ one has $u_x e_I = e_I u_x$, for $x \notin I$ one has $e_I u_x e_I = 0$.

The universal C*-algebra with these generators and relations is no longer simple but in most respects its structure is similar to the one of $\mathfrak{A}[R]$. There is a canonical maximal commutative subalgebra $\mathscr{D}$ with totally disconnected spectrum (generated by the e_I), and a Bunce-Deddens type subalgebra $\mathscr{B}$ generated by $\mathscr{D}$ together with the u_x, $x \in R$. Using this structure one shows

Theorem 4.1 (cf. [6]) *The universal C*-algebra with generators u_x, s_a, e_I satisfying the relations 1.,2.,3.,4. above is canonically isomorphic to $C^*_\lambda(R \rtimes R^\times)$. As a consequence $C^*_\lambda(R \rtimes R^\times)$ is also isomorphic to the semigroup crossed product $\mathscr{D} \rtimes (R \rtimes R^\times)$ (i.e. to the universal C*-algebra generated by $\mathscr{D}$ together with a representation of the semigroup $R \rtimes R^\times$ by isometries implementing the given endomorphisms of $\mathscr{D}$).*

It follows that $\mathfrak{A}[R]$ is a quotient of $C^*_\lambda(R \rtimes R^\times)$. As mentioned above, in the simplest case $\mathfrak{A}[R]$ is obtained from $C^*_\lambda(R \rtimes R^\times)$ by 'tightening' the relation

$$\sum_{j \in R/nR} u_j s_n s_n^* u_{-j} \leq 1$$

to $\sum_{j \in R/nR} u_j s_n s_n^* u_{-j} = 1$. This kind of tightening has occurred in many places in the literature under the name tight representation or boundary quotient etc.

The relations 1., 2., 3., 4. above turned out to also give the right framework for describing the left regular C*-algebra of more general semigroups. The theory of these regular C*-algebras has been developed by Xin Li [19, 20].

As in Sect. 3 the key to the computation of $K_*(C^*_\lambda(R \rtimes R^\times))$, for the ring R of integers in a number field K, lies in a KK-equivalence between the given action by endomorphisms of our semigroup with a basically trivial situation.

The semigroup $S = R \rtimes R^\times$ admits $G = K \rtimes K^\times$ as a canonical enveloping group. The action of S on the commutative subalgebra $\mathscr{D}$ of $C^*_\lambda(R \rtimes R^\times)$ has a natural dilation to an action of G. This means that $\mathscr{D}$ can be embedded into a larger commutative C*-algebra $\mathscr{C} \supset \mathscr{D}$ with an action of G which extends the action of S on $\mathscr{D}$ (this uses the fact that S is a directed set ordered by right divisibility). The crossed product $\mathscr{C} \rtimes G$ is then Morita equivalent to $\mathscr{D} \rtimes S \cong C^*_\lambda(R \rtimes R^\times)$ (the last isomorphism follows from Theorem 4.1).

A fractional ideal in K is a subset J of K of the form $J = aI$ where I is an ideal in R and $a \in K^\times$. Denote by $\mathscr{J}$ the set of all fractional ideals of R in K, i.e. the set of all translates in K of ideals in R under the action of $K^\times$.

It is easy to see that, in the dilated system, there is a bijection $J \mapsto e_J$ between $\mathscr{J}$ and the translates under G of the projections e_I, I ideal in R. Moreover the e_J, $J \in \mathscr{J}$ generate $\mathscr{C} \supset \mathscr{D}$. Using the fact that R is a Dedekind domain it is not difficult to show that the family $\{e_J\}$ forms a regular basis of $\mathscr{C}$ in the sense of the following definition. The importance of the regularity condition (or, in another guise, of the 'independence' of the family of constructible left ideals of the semigroup) has been noticed by Xin Li.

Definition 4.2 If $\{e_J : J \in \mathscr{J}\}$ is a countable set of non-zero projections in a commutative C*-algebra $\mathscr{C}$, we say that $\{e_J\}$ is a *regular basis* for $\mathscr{C}$ if it is linearly independent, closed under multiplication (up to 0) and generates $\mathscr{C}$ as a C*-algebra (this means that span $\{e_J : J \in \mathscr{J}\}$ is a dense subalgebra of $\mathscr{C}$).

Now the group G acts on $\mathscr{C}$, on $\mathscr{J}$ and on the algebra $\mathscr{K} = \mathscr{K}(\ell^2(\mathscr{J}))$ of compact operators. We can trivially define an equivariant *-homomorphism $\kappa : C_0(\mathscr{J}) \to \mathscr{K} \otimes \mathscr{C}$ by mapping δ_J to $\varepsilon_J \otimes e_J$. Here, δ_J denotes the indicator function of the one-point set $\{J\}$ and ε_J denotes the matrix in $\mathscr{K}$ which is 1 in the diagonal place (J, J) and 0 otherwise (matrix unit).

Theorem 4.3 ([7, 8]) *The equivariant map κ induces an isomorphism*

$$K_*(C_0(\mathscr{J}) \rtimes G) \longrightarrow K_*(\mathscr{C} \rtimes G) \cong K_*(C^*_\lambda(R \rtimes R^\times))$$

But now, by Green's imprimitivity theorem the crossed product $C_0(\mathscr{J}) \rtimes G$ is simply Morita equivalent to the direct sum, over the G-orbits in $\mathscr{J}$, of the C*-algebras of the stabilizer groups of each orbit.

Definition 4.4 The ideal class group Cl_K is the quotient of the semigroup of fractional ideals in K under the equivalence relation where J is equivalent to J' iff there is $a \in K^\times$ such that $J' = aJ$.

If R is the ring of algebraic integers in the number field K, then the class group is a finite abelian group.

Two fractional ideals J and J' are in the same orbit for G if and only if there is $a \in K^\times$ such that $J' = aJ$. Therefore, by Definition 4.4 the orbits are labeled exactly by the elements of the class group Cl_K. The stabilizer group of the class of a fractional ideal J then is given by the semidirect product $J \rtimes R^*$ of the additive group J by the group R^* of units (i.e. of invertible elements in $R^\times$). As a corollary to Theorem 4.3 we thus obtain

Corollary 4.5 *For each element γ of the class group Cl_K choose any ideal I_γ representing the class γ. Then*

$$K_*(C^*_\lambda(R \rtimes R^\times)) \cong \bigoplus_{\gamma \in Cl_K} K_*\big(C^*(I_\gamma) \rtimes R^*\big).$$

In the situation at hand, Theorem 4.3 can be proven directly—essentially in a similar way as at the end of Sect. 3 using the equivariant map κ. There is however a

much more powerful approach based on techniques from work on the Baum-Connes conjecture developed by Echterhoff and others [4, 13]. They establish the following principle:

> Assume that the group G satisfies the Baum-Connes conjecture with coefficients in the G-algebras A and B. Let $\kappa : A \to B$ be an equivariant homomorphism which induces, via descent, isomorphisms $K_*(A \rtimes H) \cong K_*(B \rtimes H)$ for all compact subgroups H of G. Then κ also induces an isomorphism $K_*(A \rtimes_r G) \cong K_*(B \rtimes_r G)$.

Theorem 4.3 then follows from checking that the equivariant map $C_0(\mathscr{J}) \to \mathscr{K} \otimes \mathscr{C}$ used there satisfies this condition for all finite subgroups of G.

This approach to Theorem 4.3 has a much broader scope of applications. It allows to extend the argument to general actions of a group G, that satisfies the Baum-Connes conjecture with coefficients, on a commutative C*-algebra $\mathscr{C}$ admitting a G-invariant regular basis of projections in the sense of Definition 4.2. In particular, it can then be used to compute the K-theory of the left regular C*-algebra for a large class of semigroups as well as for crossed products by automorphic actions by such semigroups. Moreover this more general method also allows to compute the K-theory for crossed products for an action of a group on a totally disconnected space that admits an invariant regular basis as in Definition 4.2, [7, 8].

For instance, one obtains

Theorem 4.6 ([7]) *Let R be a Dedekind domain with quotient field $Q(R)$ and A a C*-algebra. Then the following are true:*

1. *For every action $\alpha : R^\times \to \mathrm{Aut}\,(A)$ there is a canonical isomorphism*

$$K_*(A \rtimes_{\alpha,r} R^\times) \cong \bigoplus_{\gamma \in Cl_{Q(R)}} K_*(A \rtimes_{\alpha,r} R^*).$$

2. *For every action $\alpha : R^\times / R^* \to \mathrm{Aut}\,(A)$ there is a canonical isomorphism*

$$K_*(A \rtimes_{\alpha,r} (R^\times / R^*)) \cong \bigoplus_{\gamma \in Cl_{Q(R)}} K_*(A).$$

3. *For every action $\alpha : R \rtimes R^\times \to \mathrm{Aut}\,(A)$ there is a canonical isomorphism*

$$K_*\big(A \rtimes_{\alpha,r} (R \rtimes R^\times)\big) \cong \bigoplus_{\gamma \in Cl_{Q(R)}} K_*\big(A \rtimes_{\alpha,r} (I_\gamma \rtimes R^*)\big).$$

The above method of computing K-theory for semigroup C*-algebras and for certain crossed products for actions on totally disconnected spaces has been developed further by Li-Norling in [22, 23].

5 *KMS*-States

To end this survey we briefly discuss the *KMS*-structure for the natural one-parameter automorphism group of $C^*_\lambda(R \rtimes R^\times)$ where, again, R is the ring of algebraic integers in a number field K. After all, part of the motivation for the study of ring C*-algebras came from Bost-Connes systems and a main feature of such systems is the rich *KMS*-structure. Also, one of the reasons in [6] for passing from the ring C*-algebra $\mathfrak{A}[R]$ to $C^*_\lambda(R \rtimes R^\times)$ was the existence of many *KMS*-states on the latter algebra.

Recall that, for a non-zero ideal I in R, we denote by $N(I)$ the norm of I, i.e. the number $N(I) = |R/I|$ of elements in R/I. For $a \in R^\times$ we also write $N(a) = N(aR)$. The norm is multiplicative, [25]. Using the norm one defines a natural one-parameter automorphism group $(\sigma_t)_{t\in\mathbb{R}}$ on $C^*_\lambda(R \rtimes R^\times)$, given on the generators by

$$\sigma_t(u_x) = u_x \quad \sigma_t(e_I) = e_I \quad \sigma_t(s_a) = N(a)^{it} s_a$$

(this assignment manifestly respects the relations between the generators and thus induces an automorphism). Let β be a real number ≥ 0. Recall that a β-*KMS* state with respect to a one-parameter automorphism group $(\sigma_t)_{t\in\mathbb{R}}$ is a state φ which satisfies $\varphi(yx) = \varphi(x\sigma_{i\beta}(y))$ for a dense set of analytic vectors x, y and for the natural extension of (σ_t) to complex parameters on analytic vectors, [3]. For the one-parameter automorphism group σ, defined above, the β-*KMS* condition for a state φ translates to

$$\varphi(u_x z) = \varphi(z\, u_x) \quad \varphi(e_I z) = \varphi(z\, e_I) \quad \varphi(s_a z) = N(a)^{-\beta}\varphi(z\, s_a) \tag{5.1}$$

for a set of analytic vectors z with dense linear span and for the standard generators u_x, e_I, s_a of $C^*_\lambda(R \rtimes R^\times)$.

Theorem 5.1 ([6]) *The KMS-states on $C^*_\lambda(R \rtimes R^\times)$ at inverse temperature β can be described. One has*

1. *no KMS-states for $\beta < 1$.*
2. *for each $\beta \in [1, 2]$ a unique β-KMS state.*
3. *for $\beta \in (2, \infty)$ a bijection between β-KMS states and traces on*

$$\bigoplus_{\gamma\in Cl_K} C^*(I_\gamma) \rtimes R^*$$

where Cl_K is the ideal class group, I_γ is any ideal representing γ and R^* denotes the multiplicative group of invertible elements in R (units).

The simpler case of Theorem 5.1, where $R = \mathbb{Z}$, $K = \mathbb{Q}$, had essentially been treated already by Laca-Raeburn in [16]. The first assertion in Theorem 5.1 is basically obvious. The proof, in [6], of point 3. uses special representations of $C^*_\lambda(R \rtimes R^\times)$ which seem to be of independent interest. The proof of 2. in [6] uses

a result, also of some independent interest, on asymptotics of partial Dedekind ζ-functions. An alternative subsequent proof of Theorem 5.1, due to Neshveyev, is obtained by relating the problem to a general result on *KMS*-states for C*-algebras of non-principal groupoids, [24].

There is a striking parallel between the formula for the *KMS*-states for $\beta > 2$ in the theorem above and the formula for the *K*-theory of $C^*_\lambda(R \rtimes R^\times)$ in Corollary 4.5. The *K*-theory is isomorphic to the *K*-theory of the C*-algebra $\bigoplus_{\gamma\in\Gamma} C^*(I_\gamma) \rtimes R^*$ while the simplex of *KMS*-states is in bijection with the traces of this direct sum C*-algebra. Note that both results are quite non-trivial, as $\bigoplus_{\gamma\in\Gamma} C^*(I_\gamma) \rtimes R^*$ is not a natural subalgebra of $C^*_\lambda(R \rtimes R^\times)$.

Acknowledgements Research supported by DFG through CRC 878 and by ERC through AdG 267079.

References

1. Selcuk Barlak. On the spectral sequence associated with the Baum-Connes conjecture for $\mathbb{Z}^n$. *preprint*, 2015.
2. Jean-Benoit Bost and Alain Connes. Hecke algebras, type III factors and phase transitions with spontaneous symmetry breaking in number theory. *Selecta Math. (N.S.)*, 1(3):411–457, 1995.
3. Ola Bratteli and Derek W. Robinson. *Operator algebras and quantum statistical mechanics. 2*. Texts and Monographs in Physics. Springer-Verlag, Berlin, second edition, 1997. Equilibrium states. Models in quantum statistical mechanics.
4. Jerôme Chabert, Siegfried Echterhoff, and Hervé Oyono-Oyono. Going-down functors, the Künneth formula, and the Baum-Connes conjecture. *Geom. Funct. Anal.*, 14(3):491–528, 2004.
5. Joachim Cuntz. C^*-algebras associated with the $ax+b$-semigroup over $\mathbb{N}$. Cortiñas, Guillermo (ed.) et al., *K*-theory and noncommutative geometry. Proceedings of the ICM 2006 satellite conference, Valladolid, Spain, August 31–September 6, 2006. Zürich: European Mathematical Society (EMS). Series of Congress Reports, 201–215 (2008)., 2008.
6. Joachim Cuntz, Christopher Deninger, and Marcelo Laca. C^*-algebras of Toeplitz type associated with algebraic number fields. *Math. Ann.*, 355(4):1383–1423, 2013.
7. Joachim Cuntz, Siegfried Echterhoff, and Xin Li. On the *K*-theory of crossed products by automorphic semigroup actions. *Q. J. Math.*, 64(3):747–784, 2013.
8. Joachim Cuntz, Siegfried Echterhoff, and Xin Li. On the K-theory of the C*-algebra generated by the left regular representation of an Ore semigroup. *J. Eur. Math. Soc. (JEMS)*, 17(3):645–687, 2015.
9. Joachim Cuntz and Xin Li. The regular C^*-algebra of an integral domain. *Clay Mathematics Proceedings*, 10:149–170, 2010.
10. Joachim Cuntz and Xin Li. C*-algebras associated with integral domains and crossed products by actions on adele spaces. *J. Noncommut. Geom.*, 5(1):1–37, 2011.
11. Joachim Cuntz and Xin Li. K-theory for ring C*-algebras attached to polynomial rings over finite fields. *J. Noncommut. Geom.*, 5(3):331–349, 2011.
12. Joachim Cuntz and Anatoly Vershik. C^*-algebras associated with endomorphisms and polymorphsims of compact abelian groups. *Comm. Math. Phys.*, 321(1):157–179, 2013.
13. Siegfried Echterhoff, Ryszard Nest, and Hervé Oyono-Oyono. Fibrations with noncommutative fibers. *J. Noncommut. Geom.*, 3(3):377–417, 2009.

14. Ruy Exel, Astrid an Huef, and Iain Raeburn. Purely infinite simple C^*-algebras associated to integer dilation matrices. *Indiana Univ. Math. J.*, 60(3):1033–1058, 2011.
15. Ilan Hirshberg. On C^*-algebras associated to certain endomorphisms of discrete groups. *New York J. Math.*, 8:99–109 (electronic), 2002.
16. Marcelo Laca and Iain Raeburn. Phase transition on the Toeplitz algebra of the affine semigroup over the natural numbers. *Adv. Math.*, 225(2):643–688, 2010.
17. Martin Langer and Wolfgang Lück. Topological K-theory of the group C^*-algebra of a semi-direct product $\mathbb{Z}^n \rtimes \mathbb{Z}/m$ for a free conjugation action. *J. Topol. Anal.*, 4(2):121–172, 2012.
18. Xin Li. Ring C*-algebras. *Math. Ann.*, 348(4):859–898, 2010.
19. Xin Li. Semigroup C*-algebras and amenability of semigroups. *J. Funct. Anal.*, 262(10):4302–4340, 2012.
20. Xin Li. Nuclearity of semigroup C^*-algebras and the connection to amenability. *Adv. Math.*, 244:626–662, 2013.
21. Xin Li and Wolfgang Lück. K-theory for ring C*-algebras: the case of number fields with higher roots of unity. *J. Topol. Anal.*, 4(4):449–479, 2012.
22. Xin Li and Magnus Dahler Norling. Independent resolutions for totally disconnected dynamical systems I: C*-algebraic case. *to appear in J. Operator Theory*.
23. Xin Li and Magnus Dahler Norling. Independent resolutions for totally disconnected dynamical systems I: Algebraic case. *J. Algebra*, 424:98–125, 2015.
24. Sergey Neshveyev. KMS states on the C^*-algebras of non-principal groupoids. *J. Operator Theory*, 70(2):513–530, 2013.
25. Jürgen Neukirch. *Algebraic number theory*, volume 322 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999. Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder.

A New Look at C*-Simplicity and the Unique Trace Property of a Group

Uffe Haagerup

Abstract We characterize when the reduced C^*-algebra of a non-trivial group has unique tracial state, respectively, is simple, in terms of Dixmier-type properties of the group C^*-algebra. We also give a simple proof of the recent result by Breuillard, Kalantar, Kennedy and Ozawa that the reduced C^*-algebra of a group has unique tracial state if and only if the amenable radical of the group is trivial.

1 Introduction

It was shown by Murray and von Neumann that the von Neumann algebra $\mathscr{L}(G)$ of a group G is a factor (necessarily of type II_1) if and only if the group G is ICC (all non-trivial conjugacy classes are infinite). The analogous problem for the reduced group C^*-algebra $C^*_\lambda(G)$ of the group G, namely characterizing when $C^*_\lambda(G)$ is simple, respectively, when does it have a unique tracial state, turned out to be far more subtle. Powers, [11], proved in 1975 that the reduced group C^*-algebra of the free groups (or order ≥ 2) is simple and has unique tracial state. His result has since then been vastly generalized. Prompted by the observation that (a) and (b) below separately imply condition (c), see, for example [7], the question arose if the following three conditions for a group G are equivalent:

(a) $C^*_\lambda(G)$ is simple (i.e., G is C^*-simple),
(b) $C^*_\lambda(G)$ has unique tracial state (i.e., G has the unique trace property),
(c) the amenable radical of G is trivial.

Uffe Haagerup tragically passed away on July 5, 2015. The results in this article were proven by Haagerup in the early Spring of 2015, and privately communicated to his (and Magdalena Musat's) PhD student Kristian Knudsen Olesen in May 2015. Based on this, Kristian Knudsen Olesen, Magdalena Musat and Mikael Rørdam (University of Copenhagen, Denmark) have written up this paper.

U. Haagerup (✉)
Department of Mathematics and Computer Science, University of Southern Denmark, Campusvej 55, 5230 Odense M, Denmark

T.M. Carlsen et al. (eds.), *Operator Algebras and Applications*, Abel Symposia 12,
DOI 10.1007/978-3-319-39286-8_7

Kalantar and Kennedy proved in [8] that C^*-simplicity of a group is equivalent to the group having a (topologically) free boundary action. In 2014, Breuillard, Kalantar, Kennedy and Ozawa, [2], used this to prove that (b) and (c) are equivalent, and hence that (a) implies (b). The picture of the interconnections between the three properties above was finally completed very recently by Le Boudec, [1], who gave examples of groups that have the unique trace property, but are not C^*-simple. Hence (b) does not imply (a). Simple C^*-algebras in general need not have unique tracial state. For example, any metrizable Choquet simplex arises as the trace simplex of a unital simple AF-algebra, [4].

In this article we give new characterizations of when a group has the unique trace property and when it is C^*-simple in terms of (intrinsic) Dixmier-type properties of the group C^*-algebra. We also give a more direct proof of the theorem by Breuillard, Kalantar, Kennedy and Ozawa that the unique trace property is equivalent to triviality of the amenable radical of the group. This proof uses Furman's characterization (cf. [5]) of the amenable radical of a group as consisting of those elements that act trivially under any boundary action.

In the very recent preprint [9], Kennedy has independently obtained results similar to those in Theorem 4.5 of this paper, characterizing when a group is C^*-simple.

2 Boundary Actions

Recall that an action of a group[1] G on a compact Hausdorff space X is said to be *strongly proximal* if for each probability measure μ on X, the weak*-closure of the orbit $G.\mu$ contains a point-mass δ_x, for some $x \in X$. An action $G \curvearrowright X$ is said to be a *boundary action* if it is strongly proximal and minimal. If this is the case, then for each $x \in X$ and each probability measure μ on X, there is a net (s_i) in G such that $s_i.\mu$ converges to δ_x in the weak*-topology. Recall also that the *amenable radical* of G is defined to be the largest normal amenable subgroup of G, and it is denoted by $\mathrm{Rad}(G)$.

Furman, [5], proved in 2003 the following result about (the existence of) boundary actions of a group:

Theorem 2.1 (Furman) *Let G be a group and let $t \in G$. Then $t \notin \mathrm{Rad}(G)$ if and only if there is a boundary action of G on some compact Hausdorff space X such that t acts non-trivially on X.*

We denote by λ the left-regular representation of the group G on $\ell^2(G)$, and by $C^*_\lambda(G)$ the (associated) reduced group C^*-algebra. A group G is said to be C^**-simple* if $C^*_\lambda(G)$ is a simple C^*-algebra, and it is said to have the *unique trace property* if the canonical trace on $C^*_\lambda(G)$, here denoted by τ_0, is the only tracial state on $C^*_\lambda(G)$.

[1]Throughout the paper, groups are assumed to be discrete.

Kalantar and Kennedy proved in [8] that a group G is C^*-simple if and only if it has a topologically free boundary action on some compact Hausdorff space. It was observed in [2, Proposition 2.5] that the action of G on its universal boundary $\partial_F G$ of G is free if it is topologically free, and hence that the following theorem holds:

Theorem 2.2 (Breuillard–Kalantar–Kennedy–Ozawa) *Let G be a group. Then $C^*_\lambda(G)$ is simple if and only if there is a free boundary action of G on some compact Hausdorff space.*

3 Groups with the Unique Trace Property

In this section we give a new and elementary proof of one of the main theorems from [2], namely that a group has the unique trace property if and only if it has trivial amenable radical. The proof uses Theorem 2.1 by Furman, quoted above.

When a group G acts on a compact Hausdorff space X, we can then form the reduced crossed product C^*-algebra $C(X) \rtimes_r G$, see [3, Chapter 4], which in a natural way contains both $C^*_\lambda(G)$ and $C(X)$. These two subalgebras are related as follows: $\lambda(t)f\lambda(t)^* = t.f$, for all $t \in G$ and $f \in C(X)$, where $(t.f)(x) = f(t^{-1}.x)$, for $x \in X$.

Lemma 3.1 *Let G be a group acting on a compact Hausdorff space X, let $x \in X$, and let φ be a state on $C(X) \rtimes_r G$ whose restriction to $C(X)$ is the point-evaluation δ_x. Then $\varphi(\lambda(t)) = 0$, for each $t \in G$ for which $t.x \neq x$.*

Proof The assumptions in the lemma ensure that $C(X)$ is contained in the multiplicative domain of φ, see [3, Proposition 1.5.7], so

$$\varphi(\lambda(t))f(x) = \varphi(\lambda(t)f) = \varphi((t.f)\lambda(t)) = f(t^{-1}.x)\varphi(\lambda(t)),$$

for each $f \in C(X)$ and each $t \in G$. This clearly entails that $\varphi(\lambda(t)) = 0$ when $t^{-1}.x \neq x$, which again happens precisely when $t.x \neq x$. $\square$

If φ is a state on $C^*_\lambda(G)$, or on the crossed product $C(X) \rtimes_r G$, and if $t \in G$, let $t.\varphi$ denote the state given by $(t.\varphi)(a) = \varphi(\lambda(t)a\lambda(t)^*)$, where a belongs to $C^*_\lambda(G)$, respectively, to $C(X) \rtimes_r G$.

Lemma 3.2 *Let τ be a tracial state on $C^*_\lambda(G)$, let $G \curvearrowright X$ be a boundary action, and let $x \in X$. Then τ extends to a state on $C(X) \rtimes_r G$ whose restriction to $C(X)$ is point-evaluation δ_x.*

Proof Extend τ to any state ψ on $C(X) \rtimes_r G$, and let ρ be the restriction of ψ to $C(X)$. By the assumption that $G \curvearrowright X$ is a boundary action, there is a net (s_i) in G such that $s_i.\rho$ converges to δ_x in the weak*-topology. By possibly passing to a subnet we may assume that $s_i.\psi$ converges to some state φ on $C(X) \rtimes_r G$. The restriction of φ to $C(X)$ is then equal to δ_x. Moreover, since for all $s, t \in G$,

$$(s.\psi)(\lambda(t)) = \psi(\lambda(sts^{-1})) = \tau(\lambda(sts^{-1})) = \tau(\lambda(t)),$$

we see that $\varphi(\lambda(t)) = \tau(\lambda(t))$, for all $t \in G$. This shows that φ and τ agree on $C^*_\lambda(G)$. $\square$

We can now give a new and simpler proof of one of the main theorems from [2]:

Theorem 3.3 (Breuillard–Kalantar–Kennedy–Ozawa) *Let G be a group and let $t \in G$. Then $\tau(\lambda(t)) = 0$, for every tracial state τ on $C^*_\lambda(G)$, if and only if $t \notin \mathrm{Rad}(G)$. In particular, $C^*_\lambda(G)$ has a unique tracial state if and only if $\mathrm{Rad}(G)$ is trivial.*

Proof Suppose first that $t \notin \mathrm{Rad}(G)$. Then by Theorem 2.1 (Furman), there is a boundary action $G \curvearrowright X$ such that $t.x \neq x$, for some $x \in X$. Let τ be a tracial state on $C^*_\lambda(G)$. By Lemma 3.2, there is a state φ on $C(X) \rtimes_r G$ which extends τ and whose restriction to $C(X)$ is point-evaluation at x. By Lemma 3.1, it follows that $\tau(\lambda(t)) = \varphi(\lambda(t)) = 0$.

The "only if" part follows from the well-known fact, see, for example, Proposition 3 in [7] (and its proof therein), that whenever N is a normal amenable subgroup of G, then the canonical homomorphism $\mathbb{C}G \to \mathbb{C}(G/N)$ extends to a *-homomorphism $C^*_\lambda(G) \to C^*_\lambda(G/N)$. Using this fact with $N = \mathrm{Rad}(G)$, and composing the resulting *-homomorphism $C^*_\lambda(G) \to C^*_\lambda(G/\mathrm{Rad}(G))$ with the canonical trace on $C^*_\lambda(G/\mathrm{Rad}(G))$, we obtain a tracial state τ on $C^*_\lambda(G)$ which satisfies $\tau(\lambda(t)) = 1$, for all $t \in \mathrm{Rad}(G)$.

The last claim of the theorem follows from the fact that the canonical trace on $C^*_\lambda(G)$ is the unique tracial state which vanishes on $\lambda(t)$, for all $t \neq e$. $\square$

4 C^*-Simplicity and the Unique Trace Property of Groups

This section contains our main results that provide new characterizations of the unique trace property and C^*-simplicity of a group in terms of Dixmier-type properties of the group C^*-algebra.

The lemma below is well-known, see for example [10, Lemma 2.1(c)]. We include a proof for completeness.

Lemma 4.1 *Let $x, y \in C^*_\lambda(G)$ be finite positive linear combinations of elements from $\{\lambda(t) \mid t \in G\}$. Then $\|x + y\| \geq \|x\|$.*

Proof Let $\ell^2(G)_+$ denote the "positive cone" consisting of all vectors $\xi \in \ell^2(G)$ for which $\langle \xi, e_t \rangle \geq 0$, for all $t \in G$. Here $(e_t)_{t \in G}$ denotes the standard orthonormal basis of $\ell^2(G)$. It is clear that $\langle \xi, \eta \rangle \geq 0$, for all $\xi, \eta \in \ell^2(G)_+$. Moreover, each element $z \in C^*_\lambda(G)$ which is a finite positive linear combination of elements from $\{\lambda(t) \mid t \in G\}$ maps $\ell^2(G)_+$ into $\ell^2(G)_+$, and satisfies $|\langle z\xi, \eta \rangle| \leq \langle z|\xi|, |\eta| \rangle$, from which it follows that

$$\|z\| = \sup\{\langle z\xi, \eta \rangle \mid \xi, \eta \in \ell^2(G)_+,\ \|\xi\| \leq 1,\ \|\eta\| \leq 1\}.$$

The conclusion follows now easily. $\square$

Lemma 4.2 *Let G be a group and let $t \in G$. The following conditions are equivalent:*

(i). $0 \notin \overline{\operatorname{conv}}\{\lambda(sts^{-1}) \mid s \in G\}$,
(ii). $0 \notin \overline{\operatorname{conv}}\left\{\lambda(sts^{-1}) + \lambda(sts^{-1})^* \mid s \in G\right\}$,
(iii). *there exist a self-adjoint linear functional ω on $C^*_\lambda(G)$ of norm 1 and a constant $c > 0$ such that $\operatorname{Re}\omega(\lambda(sts^{-1})) \geq c$, for all $s \in G$.*

Proof For all $c_1, \dots, c_n \geq 0$ with $\sum_{k=1}^n c_k = 1$, and all $s_1, \dots, s_n \in G$ we have

$$\Big\|\sum_{k=1}^n c_k\lambda(s_kts_k^{-1})\Big\| \leq \Big\|\sum_{k=1}^n c_k\big(\lambda(s_kts_k^{-1}) + \lambda(s_kts_k^{-1})^*\big)\Big\| \leq 2\Big\|\sum_{k=1}^n c_k\lambda(s_kts_k^{-1})\Big\|.$$

The first inequality holds by Lemma 4.1, and the second by the triangle inequality. Together, the two inequalities show that (i) and (ii) are equivalent.

The fact that (ii) implies (iii) follows from a standard application of the Hahn–Banach separation theorem on the real vector space of self-adjoint elements in $C^*_\lambda(G)$, while it is clear that (iii) implies (i). □

The theorem below sharpens the first part of Theorem 3.3.

Theorem 4.3 *Let G be a group and let $t \in G$. Then $t \notin \operatorname{Rad}(G)$ if and only if*

$$0 \in \overline{\operatorname{conv}}\{\lambda(sts^{-1}) \mid s \in G\}. \tag{†}$$

Proof If (†) holds, then $\tau(\lambda(t)) = 0$, for every tracial state τ on $C^*_\lambda(G)$, whence $t \notin \operatorname{Rad}(G)$ by Theorem 3.3.

For the converse implication, suppose that (†) does not hold, and assume to reach a contradiction that $t \notin \operatorname{Rad}(G)$. By Theorem 2.1 (Furman), there is a boundary action $G \curvearrowright X$ and some $x \in X$ such that $t.x \neq x$.

By Lemma 4.2, there exist a self-adjoint linear functional ω on $C^*_\lambda(G)$ of norm 1 and a constant $c > 0$ such that $\operatorname{Re}\omega(\lambda(sts^{-1})) \geq c$, for all $s \in G$. Let $\omega = \omega_+ - \omega_-$ be the Jordan decomposition of ω, where ω_+ and ω_- are positive linear functionals with $\|\omega\| = \|\omega_+\| + \|\omega_-\|$. Observe that $\omega_+ + \omega_-$ is a state, because $\|\omega\| = 1$.

Further, extend $\omega_\pm$ to positive linear functionals $\psi_\pm$ on $C(X) \rtimes_r G$ with $\|\psi_\pm\| = \|\omega_\pm\|$. Then $\psi_+ + \psi_-$ is a state on $C(X) \rtimes_r G$ which extends the state $\omega_+ + \omega_-$, and, moreover, $\psi_+ - \psi_-$ is a self-adjoint linear functional which extends ω.

Let ρ be the restriction of $\psi_+ + \psi_-$ to $C(X)$. As in the proof of Lemma 3.2, since $G \curvearrowright X$ is a boundary action, there is a net (s_i) in G such that $s_i.\rho$ converges to the point-mass δ_x in the weak*-topology. Upon possibly passing to a subnet, we can assume that $s_i.\psi_\pm$ converge in the weak*-topology to positive functionals $\varphi_\pm$ on $C(X) \rtimes_r G$ (necessarily with the same norms as $\psi_\pm$). The restriction of $\varphi_+ + \varphi_-$ to $C(X)$ is equal to δ_x, which is a pure state on $C(X)$, so the restriction of $\varphi_\pm$ to $C(X)$ must be equal to $\|\varphi_\pm\|\cdot\delta_x$. We can now use Lemma 3.1 (applied to suitable multiples

of the positive linear functionals $\varphi_\pm$) to conclude that $\varphi_\pm(\lambda(t)) = 0$. Hence

$$\begin{aligned} 0 = \varphi_+(\lambda(t)) - \varphi_-(\lambda(t)) &= \lim_i\big(s_i.\psi_+(\lambda(t)) - s_i.\psi_-(\lambda(t))\big) \\ = \lim_i\big(s_i.\omega_+(\lambda(t)) - s_i.\omega_-(\lambda(t))\big) &= \lim_i s_i.\omega(\lambda(t)) = \lim_i \omega(\lambda(s_its_i^{-1})), \end{aligned}$$

which contradicts the fact that $\operatorname{Re}\omega(\lambda(sts^{-1})) \geq c > 0$, for all $s \in G$. □

It is well-known that groups with trivial amenable radical are ICC. This fact also follows from Theorem 4.3, since (†) can only hold for elements $t \in G$ belonging to an infinite conjugacy class.

From Theorems 4.3 and 3.3 we obtain the following:

Corollary 4.4 *Let G be a group. Then $C^*_\lambda(G)$ has a unique tracial state if and only if*

$$0 \in \overline{\operatorname{conv}}\{\lambda(sts^{-1}) \mid s \in G\},$$

for all $t \in G \setminus \{e\}$.

Using Theorem 2.2 (Breuillard–Kalantar–Kennedy–Ozawa), we can characterize C^*-simple groups as follows:

Theorem 4.5 *Let G be a group and let τ_0 denote the canonical tracial state on $C^*_\lambda(G)$. Then the following are equivalent:*

(i). *$C^*_\lambda(G)$ is simple,*
(ii). *$\tau_0 \in \overline{\{s.\varphi \mid s \in G\}}^{w^*}$, for each state φ on $C^*_\lambda(G)$,*
(iii). *$\tau_0 \in \overline{\operatorname{conv}}^{w^*}\{s.\varphi \mid s \in G\}$, for each state φ on $C^*_\lambda(G)$,*
(iv). *$\omega(1)\cdot\tau_0 \in \overline{\operatorname{conv}}^{w^*}\{s.\omega \mid s \in G\}$, for each bounded linear functional ω on $C^*_\lambda(G)$,*
(v). *for all $t_1, t_2, \ldots, t_m \in G \setminus \{e\}$,*

$$0 \in \overline{\operatorname{conv}}\left\{\lambda(s)\big(\lambda(t_1) + \lambda(t_2) + \cdots + \lambda(t_m)\big)\lambda(s)^* \mid s \in G\right\},$$

(vi). *for all $t_1, t_2, \ldots, t_m$ in $G \setminus \{e\}$ and all $\varepsilon > 0$, there exist $s_1, s_2, \ldots, s_n \in G$ such that*

$$\Big\|\sum_{k=1}^{n} \frac{1}{n}\lambda(s_kt_js_k^{-1})\Big\| < \varepsilon,$$

for $j = 1, 2, \ldots, m$.

Proof (i) ⇒ (ii). Let φ be a state on $C^*_\lambda(G)$. By Theorem 2.2 (Breuillard–Kalantar–Kennedy–Ozawa) there is a free boundary action $G \curvearrowright X$. Take any $x \in X$. Extend φ to a state ψ on $C(X) \rtimes_r G$ and let ρ be the restriction of ψ to $C(X)$. Since $G \curvearrowright X$ is a boundary action, there is a net (s_i) in G such that $s_i.\rho$ converges to the point-evaluation δ_x in the weak*-topology. Upon possibly passing to a subnet we may

assume that $s_i.\psi$ converges to some state ψ' on $C(X) \rtimes G$. Note that $s_i.\varphi$ converges to the restriction of ψ' to $C^*_\lambda(G)$.

The restriction of ψ' to $C(X)$ is δ_x, so by Lemma 3.1, together with the fact that the action of G on X is free, we deduce that $\psi'(\lambda(t)) = 0$, for all $t \in G \setminus \{e\}$. The restriction of ψ' to $C^*_\lambda(G)$ is therefore equal to τ_0. We conclude that $s_i.\varphi$ converges to τ_0.

It is trivial that (ii) implies (iii).

(iii) $\Rightarrow$ (iv). Fix states $\varphi_1, \varphi_2, \dots, \varphi_m$ on $C^*_\lambda(G)$. The set

$$\overline{\mathrm{conv}}^{w*}\{(s.\varphi_1, s.\varphi_2, \dots, s.\varphi_m) \mid s \in G\}$$

is a weak* closed G-invariant convex subset of the set of m-tuples of the set of states on $C^*_\lambda(G)$. Repeated applications of (iii) shows that the m-tuple $(\tau_0, \tau_0, \dots, \tau_0)$ belongs to this set. It follows that for each finite subset of $C^*_\lambda(G)$, and $\varepsilon > 0$, there exist $s_1, s_2, \dots, s_n \in G$ such that

$$\Big|\frac{1}{n}\sum_{k=1}^{n} s_k.\varphi_j(a) - \tau_0(a)\Big| < \varepsilon,$$

for all $a \in F$ and all $j = 1, 2, \dots, m$. Since each bounded linear functional is a linear combination of finitely many (in fact, four) states, we see that (iv) holds.

(iv) $\Rightarrow$ (v). Suppose that (v) does not hold, and let $t_1, t_2, \dots, t_m$ be elements in $G \setminus \{e\}$ that witness the failure of (v). Using Lemma 4.1 and arguing as in the proof of Lemma 4.2, we conclude that

$$0 \notin \overline{\mathrm{conv}}\Big\{\lambda(s)\big(\sum_{j=1}^{m}(\lambda(t_j) + \lambda(t_j^{-1}))\big)\lambda(s)^* \mid s \in G\Big\}.$$

By the Hahn–Banach separation theorem we obtain a self-adjoint linear functional ω on $C^*_\lambda(G)$ (of norm 1) and $c > 0$ such that for all $s \in G$,

$$2\,\mathrm{Re}\,\omega\Big(\sum_{j=1}^{n}\lambda(st_js^{-1})\Big) = \omega\Big(\sum_{j=1}^{n}\lambda(st_js^{-1}) + \lambda(st_j^{-1}s^{-1})\Big) \geq c.$$

Thus $2\,\mathrm{Re}\,\rho(\sum_{j=1}^{n}\lambda(t_j)) \geq c > 0$, for all ρ in the weak*-closure of $\mathrm{conv}\{s.\omega \mid s \in G\}$, while $\tau_0(\sum_{j=1}^{n}\lambda(t_j)) = 0$. This shows that (iv) does not hold.

(v) $\Rightarrow$ (vi). If (v) holds, then for all $t_1, t_2, \dots, t_m \in G \setminus \{e\}$ and all $\varepsilon > 0$, there are $s_1, s_2, \dots, s_n \in G$ (repetitions being allowed) such that

$$\Big\|\sum_{k=1}^{n}\frac{1}{n}\lambda(s_k)\big(\lambda(t_1) + \lambda(t_2) + \cdots + \lambda(t_m)\big)\lambda(s_k)^*\Big\| < \varepsilon.$$

Now use Lemma 4.1 to conclude that (vi) holds.

(vi) $\Rightarrow$ (i). It is easy to see that (vi) implies the Dixmier property: for each $a \in C^*_\lambda(G)$, $\overline{\text{conv}}\{uau^* \mid u \in C^*_\lambda(G) \text{ unitary}\}$ meets the scalars (necessarily at $\tau_0(a)\cdot 1$). Since τ_0 is faithful, this is easily seen to imply simplicity (and uniqueness of trace) of $C^*_\lambda(G)$, cf. [11]. $\square$

5 Summary

We end with a summary of existing results combined with results obtained in this article.

Theorem 5.1 *Let G be a group and let $t \in G$. The following are equivalent:*

(i). $t \notin \text{Rad}(G)$,
(ii). *there is a boundary action $G \curvearrowright X$ such that t acts non-trivially on X,*
(iii). $\tau(\lambda(t)) = 0$*, for all tracial states τ on $C^*_\lambda(G)$,*
(iv). $0 \in \overline{\text{conv}}\{\lambda(sts^{-1}) \mid s \in G\}$.

The equivalence of (i) and (ii) is [5, Proposition 7] by Furman (see Theorem 2.1 above), the equivalence of (ii) and (iii) is [2, Theorem 4.1] (quoted and reproved here as Theorem 3.3), and the equivalence between (iii) and (iv) is Theorem 4.3 above.

Theorem 5.2 *Let G be a group. The following are equivalent:*

(i). *$C^*_\lambda(G)$ has a unique tracial state,*
(ii). *G admits a faithful boundary action,*
(iii). $\text{Rad}(G) = \{e\}$,
(iv). *for all $t \in G \setminus \{e\}$ and all $\varepsilon > 0$, there exist $s_1, s_2, \ldots, s_n \in G$ such that*

$$\left\| \frac{1}{n} \sum_{k=1}^{n} \lambda(s_k t s_k^{-1}) \right\| < \varepsilon.$$

By universality of the Furstenberg boundary, it follows from Furman's result (cf. the equivalence of (i) and (ii) in Theorem 5.1 above) that G acts faithfully on its Furstenberg boundary if and only if the amenable radical is trivial. Hence (ii) and (iii) are equivalent.

The equivalence between (i) and (iii) is [2, Corollary 4.2] (quoted and reproved here as Theorem 3.3), while the equivalence between (iii) and (iv) is Corollary 4.4 above.

Theorem 5.3 *Let G be a group, and let τ_0 be the canonical tracial state on $C^*_\lambda(G)$. The following are equivalent:*

(i). *$C^*_\lambda(G)$ is simple,*
(ii). *G admits a free boundary action,*
(iii). $\tau_0 \in \overline{\{s.\varphi \mid s \in G\}}^{w^*}$*, for each state φ on $C^*_\lambda(G)$,*
(iv). $\tau_0 \in \overline{\text{conv}}^{w^*}\{s.\varphi \mid s \in G\}$*, for each state φ on $C^*_\lambda(G)$,*

(v). *for all* $t_1, t_2, \dots, t_m \in G \setminus \{e\}$ *and all* $\varepsilon > 0$, *there exist* $s_1, s_2, \dots, s_n \in G$ *such that*

$$\Big\| \frac{1}{n} \sum_{k=1}^{n} \lambda(s_k t_j s_k^{-1}) \Big\| < \varepsilon,$$

for $j = 1, 2, \dots, m$,

(vi). $C^*_\lambda(G)$ *has the Dixmier property, i.e.,* $\overline{\mathrm{conv}}\,\{uau^* \mid u \in C^*_\lambda(G) \text{ unitary}\} \cap \mathbb{C}\cdot 1 \neq \emptyset$, *for all* $a \in C^*_\lambda(G)$.

The equivalence of (i) and (ii) is stated as Theorem 2.2 above, and was proven by Breuillard–Kalantar–Kennedy–Ozawa. The remaining implications are contained in Theorem 4.5 (and its proof) above.

It is worth noting that in (vi) one can even take the unitaries u in $C^*_\lambda(G)$ to be in the set $\{\lambda(t) \mid t \in G\}$. It was shown in [6] that the Dixmier property holds for any unital simple C^*-algebra with at most one tracial state. Conversely, any unital C^*-algebra satisfying the Dixmier property can have at most one tracial state; moreover, it is simple if, in addition, it has a faithful trace.

It was shown in [2] as a corollary to the characterization of groups with the unique trace property therein, that simplicity of $C^*_\lambda(G)$ implies that $C^*_\lambda(G)$ has unique tracial state. This implication also follows in several different ways from the results obtained in this article. For instance, since $s.\tau = \tau$ for any tracial state τ on $C^*_\lambda(G)$ and every $s \in G$, the (equivalent) statements (iii) and (iv) in Theorem 5.3 both imply uniqueness of the trace. Respectively, the fact that Theorem 5.3 (v) clearly implies Theorem 5.2 (iv), yields yet another proof.

Finally, note that the equivalent conditions in Theorem 5.3 are strictly stronger than those in Theorem 5.1, due to the very recent results of Le Boudec, [1], showing that C^*-simplicity is not equivalent to the unique trace property.

References

1. A. Le Boudec, *C^*-simplicity and the amenable radical*, preprint (2015).
2. E. Breuillard, M. Kalantar, M. Kennedy, and N. Ozawa, *C^*-simplicity and the unique trace property for discrete groups*, preprint (2014).
3. N. P. Brown and N. Ozawa, *C^*-algebras and finite-dimensional approximations*, Graduate Studies in Mathematics, vol. 88, American Mathematical Society, Providence, RI, 2008.
4. E. G. Effros, *Dimensions and C^*-algebras*, CBMS Regional Conference Series in Mathematics, vol. 46, Conference Board of the Mathematical Sciences, Washington, D.C., 1981.
5. A. Furman, *On minimal strongly proximal actions of locally compact groups*, Israel J. Math. **136** (2003), 173–187.
6. U. Haagerup and L. Zsidó, *Sur la propriété de Dixmier pour les C^*-algèbres*, C. R. Acad. Sci. Paris Sér. I Math. **298** (1984), no. 8, 173–176.
7. P. de la Harpe, *On simplicity of reduced C^*-algebras of groups*, Bull. London Math. Soc. **39** (2007), 1–26.
8. M. Kalantar and M. Kennedy, *Boundaries of reduced C^*-algebras of discrete groups*, J. Reine Angew. Math., to appear.

9. M. Kennedy, *Characterizations of C^*-simplicity*, preprint (2015).
10. A. Y. Olshanskii and D. V. Osin, *C^*-simple groups without free subgroups*, Groups Geom. Dyn. **8** (2014), 933–983.
11. R. T. Powers, *Simplicity of the C^*-algebra associated with the free group on two generators*, Duke Math. J. **42** (1975), 151–156.

Equilibrium States on Graph Algebras

Astrid an Huef and Iain Raeburn

Abstract We consider operator-algebraic dynamical systems given by actions of the real line on unital C^*-algebras, and especially the equilibrium states (or KMS states) of such systems. We are particularly interested in systems built from the gauge action on the Toeplitz algebra and graph algebra of a finite directed graph, and we describe a complete classification of the KMS states obtained in joint work with Laca and Sims. We then discuss applications of these results to Cuntz-Pimsner algebras associated to local homeomorphisms, obtained in collaboration with Afsar. Thomsen has given bounds on the range of inverse temperatures at which KMS states may exist. We show that Thomsen's bounds are sharp.

1 Introduction

There has recently been a renewal of interest in the equilibrium states (the KMS states) of operator-algebraic dynamical systems consisting of an action of the real line (the *dynamics*) on a C^*-algebra. There has been particular interest in systems involving graph algebras and their Toeplitz extensions [3, 7, 11, 18].

Very satisfactory results have been obtained for systems associated to finite directed graphs, and we now have concrete descriptions of the simplices of KMS$_\beta$ states on the Toeplitz algebras at all inverse temperatures β [9, 10]. Here we review these results, and discuss some surprising applications to work of Thomsen on systems involving the Cuntz-Pimsner algebras of local homeomeorphisms [17]. One main conclusion of our recent work with Afsar [1] is that lower and upper bounds for the possible inverse temperatures given by Thomsen are sharp. For these applications we do not need the full strength of the general results in [10], and in this article we describe a more direct approach.

A. an Huef • I. Raeburn (✉)
Department of Mathematics and Statistics, University of Otago, PO Box 56, Dunedin 9054, New Zealand
e-mail: astrid@maths.otago.ac.nz; iraeburn@maths.otago.ac.nz

T.M. Carlsen et al. (eds.), *Operator Algebras and Applications*, Abel Symposia 12,
DOI 10.1007/978-3-319-39286-8_8

2 The Toeplitz Algebra of a Graph

We suppose that $E = (E^0, E^1, r, s)$ is a finite directed graph. A *Toeplitz-Cuntz-Krieger E-family* consists of mutually orthogonal projections $\{P_v : v \in E^0\}$ and partial isometries $\{S_e : e \in E^1\}$ in a C^*-algebra such that $S_e^* S_e = P_{s(e)}$ for every $e \in E^1$ and

$$P_v \geq \sum_{r(e)=v} S_e S_e^* \text{ for every } v \in E^0 \tag{2.1}$$

(where we interpret an empty sum as 0). Since the vertex projections are mutually orthogonal, the relation (2.1) implies that the range projections $S_e S_e^*$ are also mutually orthogonal. (See [9, Corollary 1.2], for example.)

For $n \geq 2$, we write

$$E^n := \big\{\mu = \mu_1\mu_2 \cdots \mu_n : s(\mu_i) = r(\mu_{i+1}) \text{ for } 1 \leq i < |\mu| := n\big\}$$

for the set of paths of length n in E, and note that $S_\mu := S_{\mu_1} S_{\mu_2} \cdots S_{\mu_n}$ is a partial isometry for every $\mu \in E^n$. We write $E^* := \bigcup_{n\geq 0} E^n$ for the set of finite paths. Then for $\mu, \nu, \alpha, \beta \in E^*$ we have the product formula

$$(S_\mu S_\nu^*)(S_\alpha S_\beta^*) = \begin{cases} S_{\mu\alpha'} S_\beta^* & \text{if } \alpha = \nu\alpha' \\ S_\mu S_{\beta\nu'}^* & \text{if } \nu = \alpha\nu' \\ 0 & \text{otherwise.} \end{cases} \tag{2.2}$$

The *Toeplitz algebra* $\mathscr{T}C^*(E)$ of E is the C^*-algebra generated by a universal Toeplitz-Cuntz-Krieger E-family (p, s). The product formula (2.2) implies that the elements $\{s_\mu s_\nu^* : \mu, \nu \in E^*\}$ span a $*$-subalgebra of $\mathscr{T}C^*(E)$, and hence we have

$$\mathscr{T}C^*(E) = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^*\}.$$

The quotient of $\mathscr{T}C^*(E)$ by the ideal generated by the gap projections

$$\Big\{p_v - \sum_{r(e)=v} s_e s_e^* : v \in E^0\Big\}$$

is the usual graph algebra or Cuntz-Krieger algebra $C^*(E)$.

For every graph E there is a canonical Toeplitz-Cuntz-Krieger E-family (Q, T) on the *finite-path space* $\ell^2(E^*)$, characterised by the following actions on the usual

orthonormal basis $\{h_\mu : \mu \in E^*\}$:

$$Q_v h_\mu = \begin{cases} h_\mu & \text{if } v = r(\mu) \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad T_e h_\mu = \begin{cases} h_{e\mu} & \text{if } s(e) = r(\mu) \\ 0 & \text{otherwise.} \end{cases}$$

The universal property of $\mathscr{T}C^*(E)$ then gives a representation $\pi_T = \pi_{Q,T}$ of $\mathscr{T}C^*(E)$ on $\ell^2(E^*)$ such that $\pi_T(p_v) = Q_v$ and $\pi_T(s_e) = T_e$; we call π_T the *finite-path representation*. The gap projections $Q_v - \sum_{r(e)=v} T_e T_e^*$ are the projections on $\mathbb{C}h_v$, and hence are all nonzero. Thus the uniqueness theorem for Toeplitz algebras [8, Corollary 4.2] implies that π_T is faithful.

There is a gauge action $\gamma : \mathbb{T} \to \operatorname{Aut} \mathscr{T}C^*(E)$ such that $\gamma_z(p_v) = p_v$ and $\gamma_z(s_e) = zs_e$, and this induces the usual gauge action on the quotient $C^*(E)$. We are interested in the dynamics $\alpha : \mathbb{R} \to \operatorname{Aut} \mathscr{T}C^*(E)$ given by $\alpha_t = \gamma_{e^{it}}$, and its analogue on $C^*(E)$. In particular, we wish to study the KMS states for this dynamics.

3 KMS States on the Toeplitz Algebra

The spanning elements $s_\mu s_\nu^*$ for $\mathscr{T}C^*(E)$ are all analytic for the action α. Hence if ϕ is a KMS$_\beta$ state on $(\mathscr{T}C^*(E), \alpha)$, we have

$$\begin{aligned} \phi(s_\mu s_\nu^*) &= \phi(s_\nu^* \alpha_{i\beta}(s_\mu)) = e^{-\beta|\mu|}\phi(s_\nu^* s_\mu) \\ &= e^{-\beta(|\mu|-|\nu|)}\phi(s_\mu s_\nu^*). \end{aligned}$$

So $\phi(s_\mu s_\nu^*) \neq 0 \Longrightarrow |\mu| = |\nu|$, and then

$$\phi(s_\mu s_\nu^*) \neq 0 \Longrightarrow \big(s_\nu^* s_\mu \neq 0 \text{ and } |\mu| = |\nu|\big) \Longrightarrow \mu = \nu.$$

Now a routine computation using the product formula (2.2) gives the following:

Lemma 3.1 *[9, Proposition 2.1] A state ϕ on $\mathscr{T}C^*(E)$ is KMS$_\beta$ for α if and only if*

$$\phi(s_\mu s_\nu^*) = \delta_{\mu,\nu} e^{-\beta|\mu|}\phi(p_{s(\mu)}).$$

Suppose ϕ is a KMS$_\beta$ state on $(\mathscr{T}C^*(E), \alpha)$. Then for each $v \in E^0$ the Toeplitz-Cuntz-Krieger relation gives

$$\phi(p_v) \geq \sum_{r(e)=v} \phi(s_e s_e^*) = \sum_{r(e)=v} e^{-\beta}\phi(p_{s(e)}), \tag{3.1}$$

where we interpret the empty sum as 0 if v is a source. The *vertex matrix* of E is the $E^0 \times E^0$ integer matrix A with entries

$$A(v,w) = |r^{-1}(v) \cap s^{-1}(w)|.$$

We can rewrite the inequality (3.1) as

$$e^\beta \phi(p_v) \geq \sum_{w \in E^0} \sum_{r(e)=v,\, s(e)=w} \phi(p_w) = \sum_{w\in E^0} A(v,w)\phi(p_w), \tag{3.2}$$

so $m = (m_v) := (\phi(p_v)) \in [0,\infty)^{E^0}$ satisfies $Am \leq e^\beta m$; we say that m is a *subinvariant vector* for A.

If ϕ factors through a KMS$_\beta$ state of $C^*(E)$, then we have equality throughout (3.1) and (3.2), and m satisfies $Am = e^\beta m$. If E is strongly connected, A is irreducible and e^β has to be the Perron-Frobenius eigenvalue of A. The Perron-Frobenius theorem then says many things: thus in particular we know that the eigenvalue is the spectral radius $\rho(A)$, that the eigenspace is one-dimensional, and that there is an eigenvector with positive entries (see [4, Theorem 2.6] or [16, Theorem 1.6]). Since ϕ is a state, we have

$$1 = \phi(1) = \sum_v \phi(p_v) = \sum_v m_v,$$

and hence $m = \big(\phi(p_v)\big)_{v\in E^0}$ is the unique eigenvector in $(0,\infty)^{E^0}$ with $\|m\|_1 = 1$. The formula in Lemma 3.1 says that $\phi(s_\mu s_\nu^*) = \delta_{\mu,\nu} e^{-\beta|\mu|} m_{s(\mu)}$ for all $\mu,\nu \in E^*$, so the vector m completely determines the state ϕ. Thus we recover the following elegant result of Enomoto, Fujii and Watatani [5]:

Theorem 3.2 *Suppose that E is a strongly connected graph with vertex matrix A. Then $(C^*(E),\alpha)$ has at most one KMS state. This state has inverse temperature $\ln\rho(A)$, where $\rho(A)$ is the spectral radius of A.*

They proved existence of the KMS$_{\ln\rho(A)}$ state too, but we'll get to that.

For states on $\mathscr{T}C^*(E)$, the vector m only satisfies the subinvariance relation $Am \leq e^\beta m$, but when A is irreducible Perron-Frobenius theory has things to say about this too. For example:

- If $Am \leq e^\beta m$ and $\beta = \ln\rho(A)$, then $Am = e^\beta m$, so that m is the Perron-Frobenius eigenvector.
- Suppose that $Am \leq e^\beta m$. Then $Am \neq e^\beta m \iff \beta > \ln\rho(A)$.

This suggests that we look more carefully at β larger than the *critical inverse temperature* $\beta_c := \ln\rho(A)$.

So we consider $\beta > \ln\rho(A)$. We find it interesting that, although we were motivated to do so by the Perron-Frobenius theory, which applies only when E is strongly connected, the following analysis does not require any connectivity

hypothesis on E. Thus we consider an arbitrary finite directed graph E, which could have sinks or sources, and a KMS_β state ϕ on $\mathscr{T}C^*(E)$.

Take $m = \big(\phi(p_v)\big)$ as before. Then $\epsilon := (1 - e^{-\beta}A)m$ has nonnegative entries, not all 0. Since $e^\beta > \rho(A)$, e^β is not in the spectrum of A, and $1 - e^{-\beta}A$ is invertible. Thus we can recover m as $(1 - e^{-\beta}A)^{-1}\epsilon$. Our main point is that we can describe geometrically the set of $\epsilon \in [0, 1]^{E^0}$ which arise from unit vectors m in $\ell^1(E^0)$. For $v \in E^0$, define $y^\beta \in [1, \infty)^{E^0}$ by

$$y_v^\beta := \sum_{n=0}^{\infty} \sum_{w \in E^0} e^{-\beta n} A^n(w, v) = (1 - e^{-\beta}A)^{-1}\delta_v; \tag{3.3}$$

the series converges because $\sum_n e^{-\beta n}A^n$ converges in the operator norm with sum $(1 - e^{-\beta}A)^{-1}$. Then $m := (1 - e^{-\beta}A)^{-1}\epsilon$ has $\|m\|_1 = 1$ if and only if

$$1 = \epsilon \cdot y^\beta := \sum_{v \in E^0} \epsilon_v y_v^\beta$$

(see Theorem 3.1(a) of [9]).

Then the main theorem of [9] says:

Theorem 3.3 *Suppose E is a finite graph with vertex matrix A, and $\beta > \ln \rho(A)$. Suppose $\epsilon \cdot y^\beta = 1$. Then there is a KMS_β state ϕ_ϵ of $\mathscr{T}C^*(E)$ such that*

$$\phi_\epsilon(p_v) = \big((1 - e^{-\beta}A)^{-1}\epsilon\big)_v \quad \textit{for all } v \in E^0.$$

The map $\epsilon \mapsto \phi_\epsilon$ is an isomorphism of $\Delta_\beta = \{\epsilon : \epsilon \cdot y^\beta = 1\}$ onto the simplex of KMS_β states of $(\mathscr{T}C^*(E), \alpha)$.

We proved existence of the KMS state ϕ_ϵ in [9, Theorem 3.1(b)] by a spatial argument using the finite-path representation π_T of $\mathscr{T}C^*(E)$ on $\ell^2(E^*)$. Then surjectivity of $\epsilon \mapsto \phi_\epsilon$ amounts to our earlier observation that the subinvariant vector $m = \big(\phi(p_v)\big)_{v \in E^0}$ determines a KMS state ϕ.

The set $\Delta_\beta = \{\epsilon : \epsilon \cdot y^\beta = 1\}$ parametrising the KMS_β states is a simplex in the positive cone $[0, \infty)^{E^0}$ of $\mathbb{R}^{E^0}$ with extreme points on the coordinate axes, and the vector y^β is normal to this simplex. As β decreases to the critical value $\beta_c = \ln \rho(A)$, the terms in the series on the right-hand side of (3.3) get larger, and the simplex contracts towards the origin.

The preceding analysis does not apply when $\beta = \beta_c = \ln \rho(A)$ is critical, because then the matrix $1 - e^{-\beta}A$ need not be invertible. However, we can take a sequence β_n decreasing to $\ln \rho(A)$, and use weak* compactness of the state space to get a $\mathrm{KMS}_{\ln \rho(A)}$ state of $\mathscr{T}C^*(E)$ [9, Proposition 4.1]. When E is strongly connected, this is the only $\mathrm{KMS}_{\ln \rho(A)}$ state, and we can deduce from Perron-Frobenius that it factors through the graph algebra $C^*(E)$. In particular, we recover the existence of the $\mathrm{KMS}_{\ln \rho(A)}$ state, first established by other methods in [5].

We can sum up our discussion as follows:

Corollary 3.4 *Suppose that E is a directed graph with vertex matrix A, and that $\beta \in (0, \infty)$ satisfies $\beta \geq \ln \rho(A)$. Then the map $\phi \mapsto m^\phi := \big(\phi(p_v)\big)$ is a bijection of the set of KMS$_\beta$ states of $(\mathscr{T}C^*(E), \alpha)$ onto the unit vectors m in $[0, \infty)^{E^0} \subset \ell^1(E^0)$ satisfying the subinvariance relation $Am \leq e^\beta m$.*

For $\beta > \ln \rho(A)$, Theorem 3.3 is stronger, because it describes the solutions of the subinvariance relation. But for some applications, such as those in Sect. 4, we can deal directly with the subinvariance relation in an *ad hoc* manner.

4 Dumbbell Graphs

We say that a graph E is *reducible* if it is not strongly connected, or equivalently if its vertex matrix A is not irreducible. For $v, w \in E^0$, we write $v \leq w$ to mean that

$$vE^*w := \{\mu \in E^* : r(\mu) = v \text{ and } s(\mu) = w\}$$

is nonempty (or in other words, that there is a path from w to v). Then we define a relation $\sim$ on E^0 by

$$v \sim w \Longleftrightarrow v \leq w \text{ and } w \leq v.$$

This is an equivalence relation (it is reflexive because $v \in vE^*v$), and we write $E^0/\sim$ for the set of equivalence classes.

For each $C \in E^0/\sim$, we define A_C to be the $C \times C$ matrix obtained by deleting all rows and columns involving vertices not in C. We can view A_C as the vertex matrix of the subgraph $E_C := (C, E^1 \cap r^{-1}(C) \cap s^{-1}(C), r, s)$. Each A_C is either a 1×1 zero matrix (if C is a singleton set $\{v\}$ and there is no loop at v) or an irreducible matrix (in which case we call E_C a *strongly connected component* of E). It is possible to order the set E^0 so that the vertex matrix A is block upper-triangular with diagonal blocks A_C (see [10, §2.3]), and it follows that $\rho(A) = \max_C \rho(A_C)$.

Now we consider the KMS states on $\mathscr{T}C^*(E)$ when E is reducible. There are three situations that we have to deal with:

- For $\beta > \ln \rho(A)$, Theorem 3.3 applies, and we have a $(|E^0| - 1)$-dimensional simplex of KMS$_\beta$ states on $\mathscr{T}C^*(E)$.
- For $\beta = \ln \rho(A)$, we focus on the *critical components* $C \in E^0/\sim$ that have $\rho(A_C) = \rho(A)$. The relation $\leq$ descends to a well-defined relation on the set of critical components, and then a critical component C is *minimal* if D critical and $D \leq C$ imply $D = C$. The behaviour of the KMS$_{\ln \rho(A)}$ states of $\mathscr{T}C^*(E)$ depends on the location of the minimal critical components.

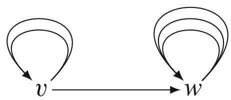
- Recall that a subset H of E^0 is *hereditary* if $v \in H$ and $v \leq w$ imply $w \in H$, and that the Toeplitz algebra $\mathscr{T}C^*(E\backslash H)$ of the graph

$$E\backslash H := (E^0\backslash H, E^1 \cap s^{-1}(E^0\backslash H), r, s)$$

is a quotient of $\mathscr{T}C^*(E)$ [10, Proposition 2.1]. For $\beta < \ln \rho(A)$, we consider the hereditary subset H of E^0 generated by the critical components (which is also generated by the minimal critical components). If H is not all of E^0, then we can apply Theorem 3.3 to $E\backslash H$ and get KMS_β states of $\mathscr{T}C^*(E)$ for $\ln \rho(A_{E\backslash H}) < \beta < \ln \rho(A)$ which factor through the quotient map onto $\mathscr{T}C^*(E\backslash H)$.

The first and third situations both require straightforward applications of Theorem 3.3, and the interesting things happen when $\beta = \ln \rho(A)$ is critical. Then the phrase "depends on the location of the minimal critical components" needs clarification. We illustrate its meaning with some examples, which fortunately are enough for the main applications in [1]. The key feature of these examples is that the strongly connected components have just one vertex each. We call such graphs *dumbbell graphs*.

Example 4.1 We consider the following graph E:



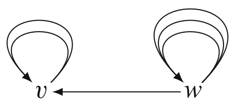
for which $\rho(A) = 3$. In this example, the hereditary closure of the critical component $\{w\}$ is all of E^0, and hence the third situation does not arise.

If we list $E^0 = \{w, v\}$, then $A = \left(\begin{smallmatrix} 3 & 1 \\ 0 & 2 \end{smallmatrix}\right)$. For $\beta = \ln \rho(A) = \ln 3$ we have $e^\beta = 3$, and the subinvariance relation $Am \leq e^\beta m$ says

$$Am = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} m_w \\ m_v \end{pmatrix} = \begin{pmatrix} 3m_w + m_v \\ 3m_v \end{pmatrix} \leq 3 \begin{pmatrix} m_w \\ m_v \end{pmatrix}.$$

The only unit vector in $[0, \infty)^{E^0} \subset \ell^1(E^0)$ which satisfies this relation is $m = (1, 0)$. Thus Corollary 3.4 says there is a unique $\mathrm{KMS}_{\ln 3}$ state on $\mathscr{T}C^*(E)$. This state factors through $C^*(E)$.

Example 4.2 Next we switch the horizontal arrow, so E is



With $E^0 = \{v, w\}$, $A = \left(\begin{smallmatrix} 2 & 1 \\ 0 & 3 \end{smallmatrix}\right)$, and subinvariance for $\beta = \ln 3$ reduces to $m_v \le m_w$. This graph also has just one critical component $\{w\}$, but this time $\{w\}$ is hereditary, and the graph $E\backslash H$ has vertex set $\{v\}$, so the third situation kicks in.

We find:

- For $\beta > \ln 3$, we have a one-dimensional simplex of KMS_β states on $\mathscr{T}C^*(E)$, none of which factor through $C^*(E)$.
- The simplex of $\mathrm{KMS}_{\ln 3}$ states on $\mathscr{T}C^*(E)$ has extreme points ϕ_v ($m_v = m_w = \frac{1}{2}$) and ϕ_w ($m_v = 0$); only ϕ_w factors through a state of $C^*(E)$.
- For $\ln 2 \le \beta < \ln 3$, there is a unique KMS_β state ϕ_v on $\mathscr{T}C^*(E)$ which factors through the quotient map corresponding to the hereditary set $\{w\} \subset E^0$.
- For $\beta = \ln 2$, the state ϕ_v factors through $C^*(E)$.
- For $\beta < \ln 2$, there are no KMS_β states on $\mathscr{T}C^*(E)$.

When a minimal strongly connected component E_C has more than one vertex, we organise the block form for A in three pieces: we take the hereditary closure H of the critical components, and decompose $E^0 = (E^0\backslash H) \cup C \cup (H\backslash C)$. The Perron-Frobenius eigenvector for A_C gives a $\mathrm{KMS}_{\ln \rho(A)}$ state ψ_C that has $\phi(p_v) = 0$ for $v \in H\backslash C$, but has $\phi(p_v) \neq 0$ for vertices v such that $vE^*C \neq \emptyset$: the precise formula is given in [10, Theorem 4.3(a)]. Since $\rho(A_{E\backslash H}) < \rho(A)$, we can also use Theorem 3.3 to find $\mathrm{KMS}_{\ln \rho(A)}$ states on $\mathscr{T}C^*(E\backslash H)$, and lift them to $\mathrm{KMS}_{\ln \rho(A)}$ states of $\mathscr{T}C^*(E)$. Again the formulas and the complete classification are given in [10, Theorem 4.3].

To construct KMS states on the usual graph algebra $C^*(E)$, we need to know which states on $\mathscr{T}C^*(E)$ factor through $C^*(E)$. Here we hit another subtlety: distinct hereditary sets give distinct ideals in $\mathscr{T}C^*(E)$ but not necessarily in $C^*(E)$, where the ideal in $C^*(E)$ associated to a hereditary subset H of E^0 depends only on the saturation of H. This problem is solved in [10, Theorem 5.3], which gives a recipe for finding all the KMS_β states of $\mathscr{T}C^*(E)$ and $C^*(E)$ for fixed β.

5 C^*-Algebras from Local Homeomorphisms

We consider a compact Hausdorff space Z and a surjective local homeomorphism $h : Z \to Z$. In our main examples in the next section, Z will be the infinite-path space E^∞ of a finite directed graph with the topology inherited from the product space $(E^1)^\infty$, and h will be the backward shift σ defined by

$$\sigma(e_1 e_2 e_3 \cdots) = e_2 e_3 \cdots .$$

If E has no sources, then σ is a homeomorphism on each cylinder set $Z(\mu)$, and hence is a local homeomorphism; if E has no sinks, then σ is also surjective. So we shall suppose in the rest of this paper that E is a finite graph with no sinks or sources, and then $\sigma : E^\infty \to E^\infty$ is a good example to bear in mind for this section.

We can view $C(Z)$ as a Hilbert bimodule X over the C^*-algebra $C(Z)$, by setting $(a \cdot x \cdot b)(z) = a(z)x(z)b(h(z))$ and

$$\langle x, y\rangle(z) = \sum_{h(w)=z} \overline{x(w)}y(w) \quad \text{for } x, y \in X.$$

This Hilbert bimodule has both a Toeplitz algebra $\mathcal{T}(X)$ and a Cuntz-Pimsner algebra $\mathcal{O}(X)$: the Toeplitz algebra is generated by a representation (ψ, π) characterised by $\psi(a \cdot x \cdot b) = \pi(a)\psi(x)\pi(b)$ and $\pi(\langle x, y\rangle) = \psi(x)^*\psi(y)$, and the Cuntz-Pimsner algebra [14] is a quotient of $\mathcal{T}(X)$. For our purposes, all the necessary background material is in Chapter 8 of [15]. The Cuntz-Pimsner algebra $\mathcal{O}(X)$ is an example of Katsura's topological-graph algebras: in the conventions of [15, Chapter 9] (which are a little different from those in Katsura's original paper [12]), the graph is (Z, Z, id, h).

The Toeplitz algebra $\mathcal{T}(X)$ carries a gauge action γ of the circle characterised by $\gamma_z(\psi(x)) = z\psi(x)$ and $\gamma_z(\pi(a)) = \pi(a)$, and this lifts to a dynamics $\alpha : t \mapsto \gamma_{e^{it}}$. The kernel of the quotient map onto $\mathcal{O}(X)$ is invariant under γ, and hence we also get a dynamics on $\mathcal{O}(X)$ (still denoted by α).

Thomsen [17] has studied the KMS states of the quotient system $(\mathcal{O}(X), \alpha)$ (and he worked with much more general systems (Z, h)). He showed that the possible inverse temperatures of the KMS states all lie in a finite interval $[\beta_l, \beta_c]$, and gave formulas for upper and lower bounds:

$$\beta_c = \limsup_{n\to\infty} \Big(n^{-1} \ln \Big(\max_{z\in Z} |h^{-n}(z)|\Big)\Big), \text{ and}$$
$$\beta_l = \limsup_{n\to\infty} \Big(n^{-1} \ln \Big(\min_{z\in Z} |h^{-n}(z)|\Big)\Big)$$

(applying [17, Theorem 6.8] with the function $F \equiv 1$; see [1, Remark 6.3] for the connections with Thomsen's notation). We are not aware that Thomsen has discussed the extent to which these bounds might be sharp.

In our recent work with Afsar [1], we have studied the KMS states of the Toeplitz system $(\mathcal{T}(X), \alpha)$. We viewed $C(Z)$ as a continuous analogue of the (finite-dimensional) space $C(E^0)$, and followed the strategy of [9]. We found that, for inverse temperatures β larger than Thomsen's β_c, the KMS$_\beta$ states are parametrised by a simplex Σ_β of finite measures ϵ on Z satisfying a normalisation condition of the form

$$\int f_\beta \, d\epsilon = 1,$$

where f_β is a fixed continuous function defined by summing a series like that defining y_β in (3.3) [1, Theorem 5.1]. At β_c, there is a phase transition: we can see by passing to limits as β decreases to β_c+ that there exist KMS$_{\beta_c}$ states on $\mathcal{T}(X)$, and can argue by mimicking our earlier results in [9] that at least one of

them factors through $\mathcal{O}(X)$. (This is Theorem 6.1 in [1].) So, in our generality at least, Thomsen's upper bound is sharp.

If E is a finite graph, then there is a natural Hilbert bimodule $X(E)$ over the commutative C^*-algebra $C(E^0)$, and the Toeplitz algebra $\mathcal{T}(X(E))$ was the original model of the Toeplitz-Cuntz-Krieger algebra $\mathcal{T}C^*(E)$ (see [8] and [15, Chapter 8]). This bimodule is not given by a local homeomorphism, so it does not quite fit the set-up of the present section, but the analysis of [1] was inspired by analogy with that of [9]. As we mentioned earlier, we can also directly apply the results of [1] to the shift σ on the compact path space E^∞, and this gives another connection to the results of [9] and [10].

6 Shifts on Path Spaces

We consider again a finite directed graph E with no sinks or sources, and the infinite-path space E^∞. Then E^∞ is a compact Hausdorff space and the backward shift $\sigma : E^\infty \to E^\infty$ is a surjective local homeomorphism. So as in Sect. 5, we can consider the Hilbert bimodule over the commutative C^*-algebra $C(E^\infty)$ with underlying space $X = C(E^\infty)$. At this point we choose to write $X(E^\infty)$ for X to emphasise that this is not the graph bimodule $X(E)$ studied in [8] and [15, Chapter 8].

The topology on E^∞ arises from viewing it as a subset of the infinite product $(E^1)^\infty$ of the finite set E^1, and the cylinder sets

$$Z(\mu) = \{x \in E^\infty : x_i = \mu_i \text{ for } i \leq |\mu|\}$$

associated to finite paths $\mu \in E^*$ form a basis of compact-open sets for the topology on E^∞. Then a straightforward calculation shows:

Lemma 6.1 *The elements $P_v := \pi(\chi_{Z(v)})$ and $S_e := \psi(\chi_{Z(e)})$ of $\mathcal{T}(X(E^\infty))$ form a Toeplitz-Cuntz-Krieger E-family.*

The universal property of the Toeplitz algebra $\mathcal{T}C^*(E)$ now gives a homomorphism $\pi_{P,S} : \mathcal{T}C^*(E) \to \mathcal{T}(X(E^\infty))$. Corollary 4.2 of [8] implies that this homomorphism is injective, and it is equivariant for the gauge actions, and hence for the various dynamics α studied in Sects. 3 and 5. So composing with $\pi_{P,S}$ takes KMS_β states of $(\mathcal{T}(X(E^\infty)), \alpha)$ to KMS_β states of $(\mathcal{T}C^*(E), \alpha)$. Now we have KMS_β states of $\mathcal{T}(X(E^\infty))$ for β larger than Thomsen's β_c, and KMS_β states of $\mathcal{T}C^*(E)$ for $\beta > \ln \rho(A)$, where A is the vertex matrix of E. We reconcile this in the following reassuring lemma, which is Proposition 7.3 of [1]. (Note that the condition on E is there to ensure that $\rho(A) > 0$, so that $\ln \rho(A)$ makes sense.)

Lemma 6.2 *Suppose that E is a directed graph with at least one cycle. Then*

$$\frac{1}{n}\ln\Big(\max_{x\in E^\infty}|\sigma^{-n}(x)|\Big)\to\ln\rho(A)\quad \textit{as } n\to\infty.$$

Thus Thomsen's β_c is our $\ln\rho(A)$, and the range of possible β in Theorem 3.3 is the same as that in [1, Theorem 5.1]. Suppose that $\beta>\ln\rho(A)$, that μ is a measure on E^∞ satisfying the hypothesis $\int f_\beta\,d\mu=1$ of [1, Theorem 5.1], and that ϕ^μ is the corresponding state of $\mathscr{T}(X(E^\infty))$. Then Proposition 7.4 of [1] says that $\phi^\mu\circ\pi_{P,S}$ is the state ϕ_ϵ of [9, Theorem 2.1] associated to the vector $\epsilon=\epsilon(\mu)=\big(\mu(Z(v))\big)$ in $[0,\infty)^{E^0}$.

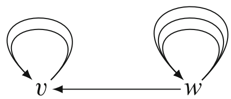
Every state ϕ_ϵ of $(\mathscr{T}C^*(E),\alpha)$ has the form $\phi^\mu\circ\pi_{P,S}$ for some measure μ on E^* satisfying $\int f_\beta\,d\mu=1$ [1, Corollary 7.6]. In the proof of this result, such a measure μ is constructed as a measure on the inverse limit $E^\infty=\varprojlim_n E^n$, and an examination of the construction shows that there is considerable leeway in building such a measure. Indeed, for each ϵ satisfying the normalisation relation $y^\beta\cdot\epsilon=1$ of [9],

$$\Big\{\lambda\in M(E^\infty)_+:\int f_\beta\,d\lambda=1\text{ and }\lambda(Z(\lambda))=\epsilon_v\text{ for }v\in E^0\Big\}$$

is a simplex of codimension $|E^0|+1$ in the cone $M(E^\infty)_+$ of positive measures. Thus there are many more KMS_β states on $\mathscr{T}(X(E^\infty))$ than on $\mathscr{T}C^*(E)$.

The injection $\pi_{P,S}:\mathscr{T}C^*(E)\to\mathscr{T}(X(E^\infty))$ is certainly not surjective—if for no other reason, because $\mathscr{T}(X(E^\infty))$ has many more KMS states. However, Proposition 7.1 of [1] says that $\pi_{P,S}$ induces an isomorphism of the Cuntz-Krieger algebra $C^*(E)$ *onto* $\mathscr{O}(X(E^\infty))$! (This observation is essentially due to Exel [6] and Brownlowe [2].) Since this isomorphism also intertwines the dynamics of [9] and that of [1], the latter algebra has effectively the same KMS states as $C^*(E)$.

We now return to the dumbbell graph E

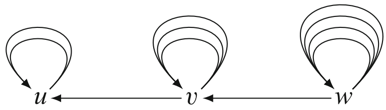


which we discussed in Example 4.2 of Sect. 4. The system $(C^*(E),\alpha)$ has KMS_β states for $\beta=\ln 3=\ln\rho(A)$ and $\beta=\ln 2=\ln\rho(A_{\{v\}})$. Thus so does $\mathscr{O}(X(E^\infty))$. We have already seen in Lemma 6.2 that $\beta_c=\ln\rho(A)$ in general. For this E and $x\in E^\infty$, we can compute

$$|\sigma^{-n}(x)|=|E^n r(x)|=\begin{cases}2^n & \text{if } r(x)=v\\ 3^n+\sum_{j=0}^{n-1}3^j2^{n-1-j} & \text{if } r(x)=w.\end{cases}$$

Thus $\min_x |\sigma^{-n}(x)| = 2^n$ is attained when $r(x) = v$, and Thomsen's β_l is $\ln 2$. So for the local homeomorphism $\sigma : E^\infty \to E^\infty$, the lower bound β_l in [17, Theorem 6.8] is also sharp.

It is easy to see with dumbbell graphs that there can be KMS_β states at inverse temperatures strictly between β_l and β_c. For example, with E the following graph



both $C^*(E)$ and $\mathcal{O}(X(E^\infty))$ have KMS states at inverse temperatures $\ln 2$, $\ln 3$ and $\ln 4$.

There are, however, interesting constraints on the possible inverse temperatures β. First, since e^β has to be the spectral radius of an irreducible integer matrix, it has to be an algebraic number. But there are also other, more subtle constraints. The issue is discussed, along with relevant results of Lind [13], in [10, §7.1].

Acknowledgements This research was supported by the Marsden Fund of the Royal Society of New Zealand. We also thank our collaborators Zahra Afsar, Marcelo Laca and Aidan Sims, the referee for some constructive suggestions, and the organisers of the Abel symposium for a marvellous experience.

References

1. Z. Afsar, A. an Huef and I. Raeburn, KMS states on C^*-algebras associated to local homeomorphisms, *Internat. J. Math.* **25** (2014), article no. 1450066 (28 pages).
2. N. Brownlowe, I. Raeburn and S.T. Vittadello, Exel's crossed product for non-unital C^*-algebras, *Math. Proc. Camb. Phil. Soc.* **149** (2010), 423–444.
3. T.M. Carlsen and N.S. Larsen, Partial actions and KMS states on relative graph C^*-algebras, arXiv:1311.0912.
4. J. Ding and A. Zhou, *Nonnegative Matrices, Positive Operators, and Applications*, World Scientific, Singapore, 2009.
5. M. Enomoto, M. Fujii and Y. Watatani, KMS states for gauge action on $\mathcal{O}_A$, *Math. Japon.* **29** (1984), 607–619.
6. R. Exel, A new look at the crossed-product of a C^*-algebra by an endomorphism, *Ergodic Theory Dynam. Systems* **23** (2003), 1733–1750.
7. R. Exel and M. Laca, Partial dynamical systems and the KMS condition, *Comm. Math. Phys.* **232** (2003), 223–277.
8. N.J. Fowler and I. Raeburn, The Toeplitz algebra of a Hilbert bimodule, *Indiana Univ. Math. J.* **48** (1999), 155–181.
9. A. an Huef, M. Laca, I. Raeburn and A. Sims, KMS states on the C^*-algebras of finite graphs, *J. Math. Anal. Appl.* **405** (2013), 388–399.
10. A. an Huef, M. Laca, I. Raeburn and A. Sims, KMS states on the C^*-algebras of reducible graphs, *Ergodic Theory Dynam. Systems* **35** (2015), 2535–2558.
11. T. Kajiwara and Y. Watatani, KMS states on finite-graph C^*-algebras, *Kyushu J. Math.* **67** (2013), 83–104.

12. T. Katsura, A class of C^*-algebras generalizing both graph algebras and homeomorphism C^*-algebras I. Fundamental results, *Trans. Amer. Math. Soc.* **356** (2004), 4287–4322.
13. D. Lind, The entropies of topological Markov shifts and a related class of algebraic integers, *Ergodic Theory Dynam. Systems* **4** (1984), 283–300.
14. M.V. Pimsner, A class of C^*-algebras generalizing both Cuntz-Krieger algebras and crossed products by $\mathbb{Z}$, *Fields Institute Comm.* **12** (1997), 189–212.
15. I. Raeburn, *Graph Algebras,* CBMS Regional Conference Series in Math., vol. 103, Amer. Math. Soc., Providence, 2005.
16. E. Senata, *Nonnegative Matrices and Markov Chains*, Second Edition, Springer-Verlag, Berlin, 1973.
17. K. Thomsen, On the C^*-algebra of a locally injective surjection, *Comm. Math. Phys.* **302** (2011), 403–423.
18. K. Thomsen, KMS weights on groupoid and graph C^*-algebras, *J. Funct. Anal.* **266** (2014), 2959–2988.

Semigroup C*-Algebras

Xin Li

Abstract This is a survey article about recent developments in semigroup C*-algebras. These C*-algebras generated by left regular representations of semigroups have been studied for some time, but it was only recently that several new connections and results were discovered, triggered by particularly interesting examples from number theory and group theory. We explain the construction of semigroup C*-algebras, introduce the basic underlying algebraic objects, define two important conditions called Toeplitz condition and independence, and present results concerning amenability and nuclearity, K-theory, and classification.

1 Introduction

The study of C*-algebras attached to semigroups was initiated by Coburn, who studied the C*-algebra of the natural numbers in [3, 4]. Motivated by index theory and related K-theoretic questions, this line of research was continued in [5, 6, 14, 15]. Later on, Murphy introduced and studied C*-algebras of more general semigroups, for instance positive cones in ordered abelian groups [35], and then also general left cancellative semigroups [36–38]. Nica studied another class of semigroup which he called positive cones in quasi-lattice ordered groups in [39]. Nica's work was taken up in [7, 8, 24].

Recently, the author tried to develop a framework which would unify all these previous studies, and used that framework to explain the connection between amenability and nuclearity in the context of semigroups and their C*-algebras (see Sect. 5, and also [25, 26]). The analysis that was necessary led to two conditions called the Toeplitz condition and the independence condition (see Sects. 3 and 4). As it turned out later, these two conditions also play a crucial role if we want to compute K-theory for semigroup C*-algebras (see Sect. 6, and also [10, 11]). Finally, putting together many of our results, we obtain classification results for

X. Li (✉)
School of Mathematical Sciences, Queen Mary University of London, Mile End Road, London E1 4NS, UK
e-mail: xin.li@qmul.ac.uk

T.M. Carlsen et al. (eds.), *Operator Algebras and Applications*, Abel Symposia 12,
DOI 10.1007/978-3-319-39286-8_9

C*-algebras attached to particular classes of semigroups (see Sect. 7, and also [17, 27, 29]).

The author's motivation to study semigroup C*-algebras did not only come from the previous work mentioned at the beginning, but also from new examples of number theoretic origin (see [9, 16, 27]). Given the importance of examples, we have included a list of examples in Sect. 2.2, and we come back to that list at various points later on in order to illustrate our results.

Topics not covered in this survey article include C*-algebras of Zappa-Szép products of semigroups [1], C*-algebras of certain topological semigroups [46, 50], or boundary quotients of semigroup C*-algebras as in [8, 49].

As this is a survey article, we have not included any proofs. Instead, we point the reader to the relevant literature. To this end, we have included an exhaustive (but not complete) list of references.

2 Semigroup C*-Algebras

We introduce C*-algebras attached to semigroups. The main objects of study are reduced semigroup C*-algebras, which are generated by left regular representations of left cancellative semigroups. Full semigroup C*-algebras are defined using left inverse hulls. We include a list of examples in order to illustrate our constructions and results.

2.1 C-Algebras Generated by Left Regular Representations*

Let P be a left cancellative semigroup. P acts on itself by left multiplication, i.e., every $p \in P$ induces a map $P \to P$, $x \mapsto px$ (we denote this map by p again). This gives rise to the left regular representation of P. To define it, consider the Hilbert space $\ell^2 P$. It comes with a canonical orthonormal basis $\{\delta_x : x \in P\}$ given by $\delta_x(y) = \delta_{x,y}$. For every $p \in P$, define an isometry V_p by setting $V_p \delta_x = \delta_{px}$. Here, our assumption that our semigroup P is left cancellative ensures that the assignment $\delta_x \mapsto \delta_{px}$ indeed extends to an isometry. The left reduced semigroup C*-algebra of P is given as follows:

Definition 2.1 $C^*_\lambda(P) := C^*\left(\{V_p : p \in P\}\right) \subseteq \mathscr{L}(\ell^2 P)$.

In a completely analogous way, we define the right reduced semigroup C*-algebra $C^*_\rho(P)$ of a right cancellative semigroup P as the C*-algebra generated by the right regular representation. Although we do not focus our discussion on this aspect, it turns out to be very interesting and intriguing to compare $C^*_\lambda(P)$ with $C^*_\rho(P)$ (for a cancellative semigroup P). We refer the interested reader to [11, 29, 34].

2.2 Examples

The first example is the additive semigroup $\mathbb{N}$ of natural numbers (including zero). In that case, $C^*_\lambda(\mathbb{N})$ is the Toeplitz algebra, the C*-algebra generated by the unilateral shift. It was studied by Coburn in [3, 4].

Here is one way to generalize our first example: Let G be a totally ordered abelian group. Denoting the order by $\geq$, we can define the positive cone in G, $P := \{x \in G : x \geq 0\}$. For instance, take an additive subgroup G of $(\mathbb{R}, +)$, view it as an ordered group with respect to the canonical total order on $\mathbb{R}$, and set $P := G \cap [0, \infty)$. Semigroup C*-algebras attached to such positive cones were studied in [5, 6, 14, 15, 28]. The original motivation came from index theory and K-theory.

Another way to generalize our first example is to look at semigroups like $\mathbb{N} \times \mathbb{N}$ or $\mathbb{N} * \mathbb{N}$ (see for instance [38] or [39]). The first one may be viewed as the semigroup (with identity) generated by two elements a and b subject to the relation $ab = ba$, while the second one is the semigroup generated by two elements without imposing any relations. From this point of view, these examples may be viewed as semigroups given by generators Σ and relations R, i.e., semigroups of the form $P = \langle \Sigma | R \rangle^+$. Of course, every semigroup may be written in that way. The point is to find classes of generators and relations which lead to well-behaved examples of semigroups. Let us mention the following particular cases:

Artin monoids are given by generators

$$\Sigma = \{\sigma_i : i \in I\},$$

where I is some (countable) index set, and relations

$$\underbrace{\sigma_i \sigma_j \ldots}_{m_{ij}} = \underbrace{\sigma_j \sigma_i \ldots}_{m_{ji}}$$

for some $m_{ij} = m_{ji} \in \{2, 3, \ldots, \infty\}$, for $i \neq j$ in I. Here $m_{ij} = m_{ji} = \infty$ means that we do not impose a relation of the form $\sigma_i \sigma_j \ldots = \sigma_j \sigma_i \ldots$ for that particular pair i, j.

A particular class of examples is given by right-angled Artin monoids, where we require $m_{ij} \in \{2, \infty\}$ for every $i \neq j$ in I. In other words, either a pair of generators commutes, or we do not impose any relation. The relations can be conveniently encoded in a graph by letting Σ be the set of vertices and connecting two vertices by an edge if the corresponding generators commute. Given an unoriented graph Γ such that every pair of vertices is joined by at most one edge, we write A_Γ^+ for the corresponding right-angled Artin monoid. The C*-algebras of such right-angled Artin monoids were studied in [7, 8, 17, 22].

Another class is given by Artin monoids of finite type. Here we require that if we add the relations $\sigma^2 = e$ for all $\sigma \in \Sigma$ to the defining relations

$$\underbrace{\sigma_i \sigma_j \ldots}_{m_{ij}} = \underbrace{\sigma_j \sigma_i \ldots}_{m_{ji}},$$

then the group generated by Σ subject to these two families of relations should become finite. Particular examples are given by braid monoids, for instance

$$B_2^+ = \langle \sigma_1, \sigma_2 | \sigma_1 \sigma_2 \sigma_1 = \sigma_2 \sigma_1 \sigma_2 \rangle^+ .$$

Another family of examples are Baumslag-Solitar monoids, which are generated by $\Sigma = \{a, b\}$ subject to the relation $ab^k = b^l a$ for some $k, l \in \mathbb{N}$. The semigroup C*-algebras of Baumslag-Solitar monoids were studied in [47] (see also [48]).

There is also the Thompson monoids, for which the generators are $\Sigma = \{x_0, x_1, \ldots\}$ and the relations are given by $x_n x_k = x_k x_{n+1}$ for $k < n$.

All these examples of semigroups given by generators and relations embed canonically into the corresponding groups defined by the same presentation. In the study of these semigroups given by generators and relations, purely algebraic considerations in the spirit of [13] or [33, §6] turn out to be very useful.

We could also consider finitely generated, abelian semigroups. Given such a semigroup P, the complex semigroup ring $\mathbb{C}[P]$ is the coordinate ring of a toric variety. A particular class of examples are given by numerical semigroups. These are of the form $P = \mathbb{N} \setminus F$, where F is a finite set such that P is still additively closed. For instance, we could take $F = \{1\}$.

As a last class of examples, we introduce $ax + b$-semigroups over integral domains. Given an integral domain R, let $P = R \rtimes R^\times$ be the semidirect product of the additive group R by the multiplicative semigroup $R^\times$ with respect to the canonical action by multiplication. A particular case is when R is the ring of algebraic integers in a number field. The corresponding semigroup C*-algebras were studied in [9, 16, 27, 29, 30].

2.3 The Left Inverse Hull, and Full Semigroup C*-Algebras

Let S be an inverse semigroup with zero 0, and set $S^\times = S \setminus \{0\}$. The reduced C*-algebra $C^*_\lambda(S)$ attached to S is defined as follows: For $s \in S$, define the partial isometry

$$\lambda_s : \ell^2 S^\times \to \ell^2 S^\times, \; \delta_x \mapsto \begin{cases} \delta_{sx} & \text{if } s^* s \geq xx^* \\ 0 & \text{else} \end{cases}.$$

The reduced C*-algebra $C^*_\lambda(S)$ of S is the sub-C*-algebra of $\mathscr{L}(\ell^2 S^\times)$ generated by $\{\lambda_s : s \in S\}$. The reader may find more information about inverse semigroups and their C*-algebras in [19–21, 40, 42].

Now assume that P is a left cancellative semigroup. We construct the left inverse hull of P as follows: Let S be the inverse semigroup of partial bijections of P generated by $p : P \xrightarrow{\cong} P$, $x \mapsto px$ $(p \in P)$. The semilattice of idempotents E of S is given by the set of constructible ideals

$$\mathscr{J} = \{p_1^{-1} q_1 \dots p_n^{-1} q_n P : p_i, q_i \in P\} \cup \{\emptyset\}.$$

As explained in [40, Corollary 3.2.13], the isometry $\ell^2 P \to \ell^2 S^\times$, $\delta_p \mapsto \delta_p$ induces a surjective homomorphism $C^*_\lambda(S) \twoheadrightarrow C^*_\lambda(P)$.

To define the full semigroup C*-algebra of P, we set $C^*(P) := C^*(S)$. By definition, $C^*(S)$ is the C*-algebra which is universal for representations of our inverse semigroup S by partial isometries, with the extra requirement that $0 \in S$ is representated by 0 in the target C*-algebra. Our definition of full semigroup C*-algebras differs from the definitions in [25, 40], where two versions of full semigroup C*-algebras (denoted by $C^*(P)$ and $C^*_s(P)$) were introduced. In particular cases, however, it is possible to identify our full semigroup C*-algebra with the one from [25, 40] (compare [40, Proposition 3.3.1 and Proposition 3.3.2]). Furthermore, we let $\lambda : C^*(P) \twoheadrightarrow C^*_\lambda(P)$ be the composite $C^*(P) = C^*(S) \twoheadrightarrow C^*_\lambda(S) \twoheadrightarrow C^*_\lambda(P)$.

3 The Toeplitz Condition

From now on, let us assume that P is a subsemigroup of a group G such that P contains the identity e. Let $D_\lambda(P) := C^*_\lambda(P) \cap \ell^\infty(P)$, where we view $\ell^\infty(P)$ as multiplication operators on $\ell^2 P$. As P is a subsemigroup of G, it turns out that we always have a partial crossed product description for $C^*_\lambda(P)$, i.e., there exists a canonically given partial action $G \curvearrowright D_\lambda(P)$ such that $C^*_\lambda(P) \cong D_\lambda(P) \rtimes_r^{\text{partial}} G$. This is already helpful. For instance, if we let Ω_P be the spectrum of $D_\lambda(P)$, and if $G \ltimes \Omega_P$ is the partial transformation groupoid attached to (the dual action corresponding to) $G \curvearrowright D_\lambda(P)$, then we get $C^*_\lambda(P) \cong C^*_r(G \ltimes \Omega_P)$, a description of our semigroup C*-algebra as a groupoid C*-algebra.

The Toeplitz condition will allow us to write $C^*_\lambda(P)$ as a crossed product attached to a global action, at least up to Morita equivalence.

Recall that S is the left inverse hull of P.

Definition 3.1 We say that $P \subseteq G$ is Toeplitz if for every $g \in G$, the partial bijection $P \cap (g^{-1}P) \to (gP) \cap P$, $x \mapsto gx$ is in S.

Let us now give an operator algebraic reformulation of the Toeplitz condition, which will immediately lead us to the desired crossed product description. Starting with our embedding $P \subseteq G$, we view $\ell^2 P$ as a subspace of $\ell^2 G$. Let $E_P \in \mathscr{L}(\ell^2 G)$

be the orthogonal projection onto $\ell^2 P$. Moreover, as $\ell^2 P$ is a subspace of $\ell^2 G$, we may view $C^*_\lambda(P)$ as a sub-C*-algebra of $\mathscr{L}(\ell^2 G)$. Furthermore, let λ be the left regular representation of G. Then we have $V_p = E_P \lambda_p E_P$ for every $p \in P$. Clearly, if we think of $\ell^\infty(G)$ as multiplication operators on $\ell^2 G$, then E_P corresponds to $1_P \in \ell^\infty(G)$. In addition, we have a canonical action $G \curvearrowright \ell^\infty(G)$. Now let $D_{P \subseteq G}$ be the smallest G-invariant sub-C*-algebra of $\ell^\infty(G)$ containing E_P. Because we have $V_p = E_P \lambda_p E_P$ for every $p \in P$, we obtain

$$C^*_\lambda(P) \subseteq E_P(D_{P \subseteq G} \rtimes_r G)E_P.$$

Moreover, E_P is a full projection in $D_{P \subseteq G} \rtimes_r G$. Therefore,

$$E_P(D_{P \subseteq G} \rtimes_r G)E_P \sim_M D_{P \subseteq G} \rtimes_r G.$$

If $C^*_\lambda(P) \supseteq E_P(D_{P \subseteq G} \rtimes_r G)E_P$, then we get desired crossed product description.

It turns out that $P \subseteq G$ is Toeplitz if and only if for every $g \in G$ with $E_P \lambda_g E_P \neq 0$, there exist $p_1, q_1, \ldots, p_n, q_n$ in P such that $E_P \lambda_g E_P = V^*_{p_1} V_{q_1} \cdots V^*_{p_n} V_{q_n}$.

In particular, if $P \subseteq G$ is Toeplitz, then $C^*_\lambda(P) = E_P(D_{P \subseteq G} \rtimes_r G)E_P$, and the latter C*-algebra is Morita equivalent to $D_{P \subseteq G} \rtimes_r G$.

For example, if P is left Ore, then P embeds into its group of left quotients $P^{-1}P$, and $P \subseteq P^{-1}P$ is Toeplitz. A left Ore semigroup is defined as follows: A semigroup is called right reversible if every pair of non-empty left ideals has a non-empty intersection. A semigroup is said to satisfy the left Ore condition if it is cancellative and right reversible. The following result can be found in [2, Theorem 1.23] or [23, §1.1]: A semigroup P can be embedded into a group G such that $G = P^{-1}P = \{q^{-1}p : p, q \in P\}$ if and only if P satisfies the left Ore condition. It is easy to check that $P \subseteq P^{-1}P$ is Toeplitz. For instance, this covers the case of cancellative, abelian semigroups. It also covers the case of $ax + b$-semigroups over integral domains.

We mention two examples of subsemigroups of groups for which the Toeplitz condition does not hold. Let $\mathbb{N} * \mathbb{N}$ be the free semigroup on two generators. We have a canonical embedding of $\mathbb{N} * \mathbb{N}$ into $\mathbb{F}_2$, the free group on two generators. It turns out that if we compose that map with the canonical quotient map $\mathbb{F}_2 \twoheadrightarrow \mathbb{F}_2/\mathbb{F}_2''$, then we still get an embedding $\mathbb{N} * \mathbb{N} \hookrightarrow \mathbb{F}_2/\mathbb{F}_2''$. This pair, however, does not satisfy the Toeplitz condition. Roughly speaking, the argument is as follows: Let $P = \mathbb{N} * \mathbb{N}$, which we view as a subsemigroup in $G = \mathbb{F}_2/\mathbb{F}_2''$. If $P \subseteq G$ were Toeplitz, then it would follow that for every $g \in G$, we must have that either $gP \cap P$ is empty or $gP \cap P$ is of the form pP for some $p \in P$. However, in our particular example, it is easy to come up with $a, b, c, d \in P$ with $ab^{-1}cd^{-1} \in \mathbb{F}_2''$ such that $e \notin ba^{-1}P$ (in G) and $b, c \in pP \Rightarrow p = e$. Then $ba^{-1}P \cap P \neq P$ and $b, c \in ba^{-1}P \cap P$, so that $ba^{-1}P \cap P$ cannot be of the form pP for some $p \in P$. Another example, which can be treated using similar ideas as our first one, is given as follows: Consider the Thompson group F, which is defined by the same generators and relations as the Thompson monoid. This means that F is the universal group generated by $\Sigma = \{x_0, x_1, \ldots\}$ subject to the relations $x_n x_k = x_k x_{n+1}$ for $k < n$. Let a and b be the canonical free

generators of $\mathbb{N}*\mathbb{N}$. It turns out that we obtain an embedding $\mathbb{N}*\mathbb{N} \hookrightarrow F$ by $a \mapsto x_0$, $b \mapsto x_1$, and that $\mathbb{N}*\mathbb{N} \hookrightarrow F$ does not satisfy the Toeplitz condition.

4 The Independence Condition

As above, we assume that P is a subsemigroup of a group G, and that P contains the identity element of G.

The independence condition is a condition on the ideal structure of our semigroup P. Recall that we have introduced the set of constructible ideals

$$\mathcal{J} = \{p_1^{-1}q_1 \dots p_n^{-1}q_n P : \ p_i, q_i \in P\} \cup \{\emptyset\}.$$

Definition 4.1 $\mathcal{J}$ is called independent if for all $X, X_1, \dots, X_n \in \mathcal{J}$, $X = \bigcup_{i=1}^n X_i$ implies $X = X_i$ for some $1 \leq i \leq n$.

We say that P satisfies the independence condition if $\mathcal{J}$ is independent.

We would like to give equivalent characterizations of independence. To do so, recall that S is the left inverse hull of P. Let E be the semilattice of idempotents of S. $\lambda : C^*(P) \twoheadrightarrow C^*_\lambda(P)$ restricts to a canonical map $C^*(E) \twoheadrightarrow D_\lambda(P)$. It was shown in [40, Theorem 3.2.14] that the following are equivalent:

- the canonical map $C^*_\lambda(S) \twoheadrightarrow C^*_\lambda(P)$ is injective,
- the canonical map $C^*(E) \twoheadrightarrow D_\lambda(P)$ is injective,
- $\mathcal{J}$ is independent.

For example, if every constructible ideal is principal, i.e., $\mathcal{J} = \{pP : p \in P\} \cup \{\emptyset\}$, then P satisfies the independence condition. This covers the case of positive cones in quasi-lattice ordered groups (see [26, 39]). Moreover, $ax + b$-semigroups over Dedekind domains, or more generally, Krull rings, satisfy the independence condition (see [25, 29]).

Let us also present two (classes of) counterexamples, i.e., semigroups which do not satisfy the independence condition. For instance, consider the numerical semigroup $\mathbb{N} \setminus \{1\}$. It turns out that it does not satisfy independence. More generally, every numerical semigroup of the form $\mathbb{N} \setminus F$, where F is a non-empty, finite subset of $\mathbb{N}$, does not satisfy independence. Another class of examples is given as follows: For certain integral domains which are not Krull rings, their multiplicative semigroups or $ax + b$-semigroups do not satisfy independence. For example, $R = \mathbb{Z}[i\sqrt{3}]$ is such an example. For more details, we refer the reader to [29, §7].

5 Amenability and Nuclearity

Let P be a subsemigroup of a group G. The following result explains the connection between amenability (of underlying dynamical systems) and nuclearity for semigroup C*-algebras.

Theorem 5.1 *Consider the following statements:*

(i) $C^*(P)$ *is nuclear,*
(ii) $C_\lambda^*(P)$ *is nuclear,*
(iii) $G \ltimes \Omega_P$ *is amenable,*
(iv) $\lambda : C^*(P) \twoheadrightarrow C_\lambda^*(P)$ *is an isomorphism.*

We always have (i) $\Rightarrow$ *(ii)* $\Leftrightarrow$ *(iii). If* P *satisfies the independence condition, then (iii)* $\Rightarrow$ *(iv) and (iii)* $\Rightarrow$ *(i), so that (i), (ii) and (iii) are equivalent.*

Here $G \ltimes \Omega_P$ is the partial translation groupoid from Sect. 3. A proof of this theorem will appear in [32], but the reader may also consult [26] for a special case of that theorem.

Moreover, it was observed in [40], and also follows immediately from the partial crossed product description discussed in Sect. 3 together with [18] or [21, Theorem 20.7 and Theorem 25.10] that if P is a subsemigroup of an amenable group, then $C_\lambda^*(P)$ is nuclear.

To give a particular class of examples, it was shown in [22], and also in [26], that semigroup C*-algebras attached to right-angled Artin monoids are nuclear.

6 K-Theory

Let P be a subsemigroup of a group G. The following result gives a formula for the K-theory of semigroup C*-algebras.

Theorem 6.1 *Assume that* P *satisfies the independence condition, and that* $P \subseteq G$ *is Toeplitz. If* G *satisfies the Baum-Connes conjecture with coefficients, then*

$$K_*(C_\lambda^*(P)) \cong \bigoplus_{[X] \in G \backslash (\mathcal{J}_{P \subseteq G} \backslash \{\emptyset\})} K_*(C_r^*(G_X)).$$

Here $\mathcal{J}_{P \subseteq G} := \{\bigcap_{i=1}^n g_i P : g_1, \ldots, g_n \in G\} \cup \{\emptyset\}$, and $G_X = \{g \in G : gX = X\}$.

We refer the reader to [10, 11] for the proof of this theorem as well as further details concerning K-theory for semigroup C*-algebras.

Let us apply our general K-theoretic formula to $ax + b$-semigroups over integral domains. Let R be a countable Krull ring with group of multiplicative units R^* and divisor class group $C(R)$. For every $\mathfrak{k} \in C(R)$, let $\mathfrak{a}_\mathfrak{k}$ be a divisorial ideal which

represents $\mathfrak{k}$. Then

$$K_*(C^*_\lambda(R \rtimes R^\times)) \cong \bigoplus_{\mathfrak{k} \in C(R)} K_*(C^*(\mathfrak{a}_\mathfrak{k} \rtimes R^*)).$$

The reader may consult [29] for more details. In particular, if R is the ring of algebraic integers in a number field K, we get

$$K_*(C^*_\lambda(R \rtimes R^\times)) \cong \bigoplus_{\mathfrak{k} \in Cl_K} K_*(C^*(\mathfrak{a}_\mathfrak{k} \rtimes R^*)).$$

Here Cl_K is the class group of K, and R^* is the group of invertible elements in $R^\times$ (i.e., the units in R or K).

If P is the positive cone in a totally ordered group G, then it is easy to see that P satisfies the independence condition, and also that $P \subseteq G$ is Toeplitz. Moreover, in that case, we have $\mathcal{J}_{P \subseteq G} = \{gP : \, g \in G\} \cup \{\emptyset\}$, and $G_P = P^* = \{e\}$. Therefore, if G satisfies the Baum-Connes conjecture with coefficients, then we get in K-theory

$$K_*(C^*_\lambda(P)) \cong K_*(\mathbb{C}).$$

The ideas in [10, 11] have been taken up in [34, 41] in order to obtain more general K-theoretic computations which go beyond semigroup C*-algebras.

7 Classification Results

We start with a classification result for semigroup C*-algebras attached to right-angled Artin monoids. We need to introduce the following terminology: A graph $\Gamma = (V, E)$ is called co-reducible if there exist non-empty subsets V_1 and V_2 of V with $V = V_1 \sqcup V_2$ such that $V_1 \times V_2 \subseteq E$. Γ is called co-irreducible if Γ is not co-reducible. In general, we can always decompose Γ into co-irreducible components, i.e., co-irreducible subgraphs $\Gamma_i = (V_i, E_i)$. Let $t(\Gamma)$ be the number of those co-irreducible components which are singletons. Moreover, let us define the Euler characteristic of a graph $\Gamma' = (V', E')$. To this end, we view Γ' as a simplicial complex by defining for every $n = 0, 1, 2, \ldots$ the set of n-simplices by

$$K_n := \{\{v_0, \ldots, v_n\} \subseteq V' : \, (v_i, v_j) \in E' \text{ for all } i, j \in \{0, \ldots, n\}, \, i \neq j\}.$$

Then we set $\chi(\Gamma') := 1 - \sum_{n=0}^{\infty} (-1)^n |K_n|$. Given a graph Γ and $k \in \mathbb{Z}$, let $N_k(\Gamma)$ be the number of co-irreducible components of Γ with Euler characteristic equal to k.

Theorem 7.1 *Let Γ and Λ be finite graphs. The following are equivalent:*

1. $C^*_\lambda(A^+_\Gamma) \cong C^*_\lambda(A^+_\Lambda)$
2. a. $t(\Gamma) = t(\Lambda)$
   b. $N_k(\Gamma) + N_{-k}(\Gamma) = N_k(\Lambda) + N_{-k}(\Lambda)$ *for all* $k \in \mathbb{Z}$
   c. $N_0(\Gamma) > 0$ *or*

$$\sum_{k>0} N_k(\Gamma) \equiv \sum_{k>0} N_k(\Lambda) \mod 2.$$

We remark that it is not necessary to restrict to finite graphs, and refer to [17] for the general result and further details.

Here is another classification result, this time for semigroup C*-algebras attached to $ax+b$-semigroups over rings of algebraic integers in number fields.

Theorem 7.2 *Let K and L be number fields with rings of algebraic integers R and S. Assume that K and L have the same number of roots of unity. If $C^*_\lambda(R \rtimes R^\times) \cong C^*_\lambda(S \rtimes S^\times)$ then K and L are arithmetically equivalent, i.e., $\zeta_K = \zeta_L$.*

Assume, in addition, that K and L are Galois extensions. In that case, we have $C^*_\lambda(R \rtimes R^\times) \cong C^*_\lambda(S \rtimes S^\times)$ if and only if $K \cong L$.

The reader may find the proof, and further information, in [27]. [43, 44] contain more details about arithmetic equivalence.

If we ask for isomorphism of semigroup C*-algebras preserving the canonical sub-C*-algebras, then we get a stronger conclusion:

Theorem 7.3 *Let K and L be number fields with rings of algebraic integers R and S. If there exists an isomorphism $C^*_\lambda(R \rtimes R^\times) \cong C^*_\lambda(S \rtimes S^\times)$ which restricts to an isomorphism $D_\lambda(R \rtimes R^\times) \cong D_\lambda(S \rtimes S^\times)$, then K and L are arithmetically equivalent and $Cl_K \cong Cl_L$ (as groups).*

Here D_λ denotes the sub-C*-algebra defined in Sect. 3. This result is proven in [30]. Our conclusion is really stronger than the one in the previous theorem, see [12].

Such sub-C*-algebras like D_λ are Cartan subalgebras in the C*-algebraic sense (see [45]). They are studied in a general context in [31].

Furthermore, we mention that there are more classification results for $ax+b$-semigroups over more general integral domains in [29].

For Baumslag-Solitar monoids, Spielberg obtained classification results in [47].

Acknowledgements Research supported by EPSRC grant EP/M009718/1.

References

1. N. Brownlowe, J. Ramagge, D. Robertson, M.F. Whittaker, *Zappa-Szép products of semigroups and their C*-algebras*, J. Funct. Anal. *226* (2014), 3937–3967.
2. A.H. Clifford and G.B. Preston, *The algebraic theory of semigroups*, vol. I, Mathematical Surveys, Number 7, American Mathematical Society, Providence, RI, 1961.
3. L.A. Coburn, *The C*-algebra generated by an isometry I*, Bull. Amer. Math. Soc. *73* (1967), 722–726.
4. L.A. Coburn, *The C*-algebra generated by an isometry II*, Trans. Amer. Math. Soc. *137* (1969), 211–217.
5. L.A. Coburn and R.G. Douglas, *C*-algebras of operators on a half-space*, Publ. Math., Inst. Hautes Étud. Sci *40* (1971), 59–68.
6. L.A. Coburn, R.G. Douglas, D. G. Schaeffer and I. M. Singer, *C*-algebras of operators on a half-space II. Index theory*, Publ. Math., Inst. Hautes Étud. Sci *40* (1971), 69–79.
7. J. Crisp and M. Laca, *On the Toeplitz algebras of right-angled and finite-type Artin groups*, J. Austral. Math. Soc. *72* (2002), 223–245.
8. J. Crisp and M. Laca, *Boundary quotients and ideals of Toeplitz algebras of Artin groups*, J. Funct. Anal. *242* (2007), 127–156.
9. J. Cuntz, C. Deninger and M. Laca, *C*-algebras of Toeplitz type associated with algebraic number fields*, Math. Ann. *355* (2013), 1383–1423.
10. J. Cuntz, S. Echterhoff and X. Li, *On the K-theory of the C*-algebra generated by the left regular representation of an Ore semigroup*, J. Eur. Math. Soc. *17* (2015), 645–687.
11. J. Cuntz, S. Echterhoff and X. Li, *On the K-theory of crossed products by automorphic semigroup actions*, Quart. J. Math. 64 (2013), 747–784.
12. B. de Smit and R. Perlis, *Zeta functions do not determine class numbers*, Bulletin AMS (New Series) *31* (1994), 213–215.
13. P. Dehornoy, *Complete positive group presentations*, Journal of Algebra *268* (2003), 156–197.
14. R.G. Douglas, *On the C*-algebra of a one-parameter semigroup of isometries*, Acta Math. *128* (1972), 143–151.
15. R.G. Douglas and R. Howe, *On the C*-algebra of Toeplitz operators on the quarterplane*, Trans. Amer. Math. Soc. *158* (1971), 203–217.
16. S. Echterhoff and M. Laca, *The primitive ideal space of the C*-algebra of the affine semigroup of algebraic integers*, Math. Proc. Camb. Phil. Soc. *154* (2013), 119–126.
17. S. Eilers, X. Li and E. Ruiz, *The isomorphism problem for semigroup C*-Algebras of right-angled Artin monoids*, Documenta Math. *21* (2016), 309–343.
18. R. Exel, *Amenability for Fell bundles*, J. Reine Angew. Math. *492* (1997), 41–73.
19. R. Exel, *Inverse semigroups and combinatorial C*-algebras*, Bull. Braz. Math. Soc. (N.S.) *39* (2008), 191–313.
20. R. Exel, *Tight representations of semilattices and inverse semigroups*, Semigroup Forum *79* (2009), 159–182.
21. R. Exel, *Partial Dynamical Systems, Fell Bundles and Applications*, preprint, arXiv:1511.04565.
22. N. Ivanov, *The K-theory of Toeplitz C*-algebras of right-angled Artin groups*, Trans. Amer. Math. Soc. *362* (2010), 6003–6027.
23. M. Laca, *From endomorphisms to automorphisms and back: dilations and full corners*, J. London Math. Soc. *61* (2000), 893–904.
24. M. Laca, I. Raeburn, *Semigroup Crossed Products and the Toeplitz Algebras of Nonabelian Groups*, J. Functional Analysis *139* (1996), 415–440.
25. X. Li, *Semigroup C*-algebras and amenability of semigroups*, J. Functional Analysis *262* (2012), 4302–4340.

26. X. LI, *Nuclearity of semigroup C*-algebras and the connection to amenability*, Adv. in Math. *244* (2013), 626–662.
27. X. LI, *On K-theoretic invariants of semigroup C*-algebras attached to number fields*, Advances in Mathematics *264* (2014), 371–395.
28. X. LI, *A new approach to recent constructions of C*-algebras from modular index theory*, J. Funct. Anal. *269* (2015), 841–864.
29. X. LI, *Semigroup C*-algebras for $ax + b$-semigroups*, Trans. Amer. Math. Soc. *368* (2016), 4417–4437.
30. X. LI, *On K-theoretic invariants of semigroup C*-algebras attached to number fields, Part II*, Adv. in Math. *291* (2016), 1–11.
31. X. LI, *Continuous orbit equivalence rigidity*, preprint, arXiv:1503.01708.
32. X. LI, *Partial transformation groupoids attached to graphs and semigroups*, preprint, arXiv:1603.09165.
33. X. LI and M. D. NORLING, *Independent resolutions for totally disconnected dynamical systems I: Algebraic case*, J. Alg. *424* (2015), 98–125.
34. X. LI and M.D. NORLING, *Independent resolutions for totally disconnected dynamical systems II: C*-algebraic case*, J. Op. Th. *75* (2016), 163–193.
35. G.J. MURPHY, *Ordered groups and Toeplitz algebras*, J. Op. Th. *18* (1987), 303–326.
36. G.J. MURPHY, *Ordered groups and crossed products of C*-algebras*, Pacific J. Math. *2* (1991), 319–349.
37. G.J. MURPHY, *Crossed products of C*-algebras by semigroups of automorphisms*, Proc. London Math. Soc. *3* (1994), 423–448.
38. G.J. MURPHY, *C*-algebras generated by commuting isometries*, Rocky Mountain J. Math. *26* (1996), 237–267.
39. A. NICA, *C*-algebras generated by isometries and Wiener-Hopf operators*, J. Operator Theory *27* (1992), 17–52.
40. M.D. NORLING, *Inverse semigroup C*-algebras associated with left cancellative semigroups*, Proc. Edinb. Math. Soc. (Series 2) *57* (2014), 533–564.
41. M.D. NORLING, *The K-theory of some reduced inverse semigroup C*-algebras*, Math. Scand. *117* (2015), 186–202.
42. A.T. PATERSON, *Groupoids, inverse semigroups, and their operator algebras*, Birkhäuser, Boston, 1999.
43. R. PERLIS, *On the equation $\zeta_K(s) = \zeta_{K'}(s)$*, J. Number Th. *9* (1977), 342–360.
44. R. PERLIS and D. STUART, *A new characterization of arithmetic equivalence*, J. Number Th. *53* (1995), 300–308.
45. J. RENAULT, *Cartan subalgebras in C*-algebras*, Irish Math. Soc. Bull. *61* (2008), 29–63.
46. J. RENAULT and S. SUNDAR, *Groupoids associated to Ore semigroup actions*, J. Op. Th. *73* (2015), 491–514.
47. J. SPIELBERG, *C*-algebras for categories of paths associated to the Baumslag-Solitar groups*, J. London Math. Soc. *86* (2012), 728–754.
48. J. SPIELBERG, *Groupoids and C*-algebras for categories of paths*, Trans. Amer. Math. Soc. *366* (2014), 5771–5819.
49. C. STARLING, *Boundary quotients of C*-algebras of right LCM semigroups*, J. Funct. Anal. *268* (2015), 3326–3356.
50. S. SUNDAR, *C*-algebras associated to topological Ore semigroups*, to appear in Muenster J. Math., arXiv:1408.4242.

Topological Full Groups of Étale Groupoids

Hiroki Matui

Abstract This is a survey of the recent development of the study of topological full groups of étale groupoids on the Cantor set. Étale groupoids arise from dynamical systems, e.g. actions of countable discrete groups, equivalence relations. Minimal $\mathbb{Z}$-actions, minimal $\mathbb{Z}^N$-actions and one-sided shifts of finite type are basic examples. We are interested in algebraic, geometric and analytic properties of topological full groups. More concretely, we discuss simplicity of commutator subgroups, abelianization, finite generation, cohomological finiteness properties, amenability, the Haagerup property, and so on. Homology groups of étale groupoids, groupoid C^*-algebras and their K-groups are also investigated.

1 Introduction

We discuss various properties of topological full groups of topological dynamical systems on Cantor sets. The study of full groups in the setting of topological dynamics was initiated by Giordano, Putnam and Skau [15]. For a minimal action $\varphi : \mathbb{Z} \curvearrowright X$ on a Cantor set X, they defined several types of full groups and showed that these groups completely determine the orbit equivalence class, the strong orbit equivalence class and the flip conjugacy class of φ, respectively.

The notion of topological full groups was later generalized to the setting of essentially principal étale groupoids $\mathscr{G}$ on Cantor sets in [27]. Étale groupoids (called r-discrete groupoids in [35]) provide us a natural framework for unified treatment of various topological dynamical systems. The topological full group $[[\mathscr{G}]]$ of $\mathscr{G}$ is a subgroup of $\mathrm{Homeo}(\mathscr{G}^{(0)})$ consisting of all homeomorphisms of $\mathscr{G}^{(0)}$ whose graph is 'contained' in the groupoid $\mathscr{G}$ as a compact open subset (see Definition 4.1). From an action φ of a discrete group Γ on a Cantor set X, we can construct the étale groupoid $\mathscr{G}_\varphi$, which is called the transformation groupoid (see Example 2.3). The topological full group $[[\mathscr{G}_\varphi]]$ of $\mathscr{G}_\varphi$ is the group of $\alpha \in \mathrm{Homeo}(X)$ for which there exists a continuous map $c : X \to \Gamma$ such that $\alpha(x) = \varphi^{c(x)}(x)$ for all

H. Matui (✉)
Graduate School of Science, Chiba University, Inage-ku, Chiba 263-8522, Japan
e-mail: matui@math.s.chiba-u.ac.jp

T.M. Carlsen et al. (eds.), *Operator Algebras and Applications*, Abel Symposia 12,
DOI 10.1007/978-3-319-39286-8_10

$x \in X$. Many other examples of étale groupoids and topological full groups will be provided in later sections.

One of the most fundamental result for topological full groups is the isomorphism theorem (Theorem 5.1), which says that $\mathcal{G}_1$ is isomorphic to $\mathcal{G}_2$ if and only if $[[\mathcal{G}_1]]$ is isomorphic to $[[\mathcal{G}_2]]$. In general, it is often difficult to distinguish two discrete groups. But, the étale groupoids have rich information about the topological dynamical systems, and so the isomorphism theorem helps us to determine the isomorphism class of the topological full groups.

The homology groups $H_n(\mathcal{G})$ for $n \geq 0$ are defined for étale groupoids $\mathcal{G}$ (see Definition 3.1). When $\mathcal{G}$ is a transformation groupoid $\mathcal{G}_\varphi$, the homology $H_n(\mathcal{G}_\varphi)$ agrees with the group homology (Example 3.3 (2)). In many examples, we can check that the homology groups 'coincide' with the K-groups of $C_r^*(\mathcal{G})$. Thus, we have isomorphisms $\bigoplus_n H_{2n+i}(\mathcal{G}) \cong K_i(C_r^*(\mathcal{G}))$ for $i = 0, 1$. This phenomenon is formulated as the HK conjecture (Conjecture 3.5).

In many cases, it is known that the commutator subgroup $D([[\mathcal{G}]])$ of $[[\mathcal{G}]]$ becomes simple (Theorem 6.5 (1), Theorem 7.3 (1)). So, it is natural to consider the abelianization $[[\mathcal{G}]]_{\mathrm{ab}} = [[\mathcal{G}]]/D([[\mathcal{G}]])$. It turns out that the abelian group $[[\mathcal{G}]]_{\mathrm{ab}}$ is closely related to the homology groups of $\mathcal{G}$. This relation is formulated as the AH conjecture (Conjecture 4.7).

In addition to these two conjectures, we are interested in several properties of $[[\mathcal{G}]]$. In [26], it was shown that $D([[\mathcal{G}_\varphi]])$ is finitely generated if $\varphi : \mathbb{Z} \curvearrowright X$ is a minimal subshift (see Theorem 8.3 (1)). In [28], it was shown that, for any SFT groupoid $\mathcal{G}_A$ (see Example 2.5), $[[\mathcal{G}_A]]$ is of type F_∞ and $D([[\mathcal{G}_A]])$ is finitely generated (Theorem 8.8). Such finiteness conditions of topological full groups are important problems. In [20], it was shown that, for any minimal action $\varphi : \mathbb{Z} \curvearrowright X$, $[[\mathcal{G}_\varphi]]$ is amenable (see Theorem 8.4). In [28], it was shown that $[[\mathcal{G}_A]]$ has the Haagerup property for any SFT groupoid $\mathcal{G}_A$ (Theorem 8.9). Such analytic properties of topological full groups are also our main concern.

2 Preliminaries

2.1 Étale Groupoids

The cardinality of a set A is written $\#A$ and the characteristic function of A is written 1_A. The finite cyclic group of order n is denoted by $\mathbb{Z}_n = \{\bar{r} \mid r = 1, 2, \dots, n\}$. We say that a subset of a topological space is clopen if it is both closed and open. A topological space is said to be totally disconnected if its topology is generated by clopen subsets. By a Cantor set, we mean a compact, metrizable, totally disconnected space with no isolated points. It is known that any two such spaces are homeomorphic. The homeomorphism group of a topological space X is written Homeo(X). The commutator subgroup of a group Γ is denoted by $D(\Gamma)$. We let Γ_{ab} denote the abelianization $\Gamma/D(\Gamma)$.

In this article, by an étale groupoid we mean a second countable locally compact Hausdorff groupoid such that the range map is a local homeomorphism. We refer the reader to [35, 36] for background material on étale groupoids. Roughly speaking, a groupoid $\mathscr{G}$ is a 'group-like' object, in which the product may not be defined for all pairs in $\mathscr{G}$. An étale groupoid $\mathscr{G}$ is equipped with locally compact Hausdorff topology, which is compatible with the groupoid structure, and the map $g \mapsto gg^{-1}$ is a local homeomorphism. For an étale groupoid $\mathscr{G}$, we let $\mathscr{G}^{(0)}$ denote the unit space and let s and r denote the source and range maps, i.e. $s(g) = g^{-1}g$ and $r(g) = gg^{-1}$. An element $g \in \mathscr{G}$ can be thought of as an arrow from $s(g)$ to $r(g)$. For $x \in \mathscr{G}^{(0)}$, $\mathscr{G}(x) = r(\mathscr{G}x)$ is called the $\mathscr{G}$-orbit of x. When every $\mathscr{G}$-orbit is dense in $\mathscr{G}^{(0)}$, $\mathscr{G}$ is said to be minimal. For a subset $Y \subset \mathscr{G}^{(0)}$, the reduction of $\mathscr{G}$ to Y is $r^{-1}(Y) \cap s^{-1}(Y)$ and denoted by $\mathscr{G}|Y$. If Y is clopen, then the reduction $\mathscr{G}|Y$ is an étale subgroupoid of $\mathscr{G}$ in an obvious way. For $x \in \mathscr{G}^{(0)}$, we write $\mathscr{G}_x = r^{-1}(x) \cap s^{-1}(x)$ and call it the isotropy group of x. The isotropy bundle of $\mathscr{G}$ is $\mathscr{G}' = \{g \in \mathscr{G} \mid r(g) = s(g)\} = \bigcup_{x \in \mathscr{G}^{(0)}} \mathscr{G}_x$. We say that $\mathscr{G}$ is principal if $\mathscr{G}' = \mathscr{G}^{(0)}$. When the interior of $\mathscr{G}'$ is $\mathscr{G}^{(0)}$, we say that $\mathscr{G}$ is essentially principal.

A subset $U \subset \mathscr{G}$ is called a $\mathscr{G}$-set if $r|U, s|U$ are injective. Any open $\mathscr{G}$-set U induces the homeomorphism $(r|U) \circ (s|U)^{-1}$ from $s(U)$ to $r(U)$. We write $\theta(U) = (r|U) \circ (s|U)^{-1}$. When U, V are $\mathscr{G}$-sets,

$$U^{-1} = \{g \in \mathscr{G} \mid g^{-1} \in U\}$$

and

$$UV = \{gg' \in \mathscr{G} \mid g \in U,\ g' \in V,\ s(g) = r(g')\}$$

are also $\mathscr{G}$-sets. A probability measure μ on $\mathscr{G}^{(0)}$ is said to be $\mathscr{G}$-invariant if $\mu(r(U)) = \mu(s(U))$ holds for every open $\mathscr{G}$-set U. The set of all $\mathscr{G}$-invariant probability measures is denoted by $M(\mathscr{G})$.

For an étale groupoid $\mathscr{G}$, we denote the reduced groupoid C^*-algebra of $\mathscr{G}$ by $C_r^*(\mathscr{G})$ and identify $C_0(\mathscr{G}^{(0)})$ with a subalgebra of $C_r^*(\mathscr{G})$. J. Renault obtained the following theorem (see also [27, Theorem 5.1]).

Theorem 2.1 ([36, Theorem 5.9]) *Two essentially principal étale groupoids $\mathscr{G}_1$ and $\mathscr{G}_2$ are isomorphic if and only if there exists an isomorphism $\varphi : C_r^*(\mathscr{G}_1) \to C_r^*(\mathscr{G}_2)$ such that $\varphi(C_0(\mathscr{G}_1^{(0)})) = C_0(\mathscr{G}_2^{(0)})$.*

2.2 Examples

In this subsection, we present several examples of étale groupoids. Throughout this subsection, by an étale groupoid, we mean a second countable étale groupoid whose unit space is the Cantor set.

Example 2.2 (AF Groupoids) We would like to recall the notion of AF groupoids ([35, Definition III.1.1], [16, Definition 3.7], [27, Definition 2.2]). Let $\mathcal{G}$ be an étale groupoid.

- We say that $\mathcal{K} \subset \mathcal{G}$ is an elementary subgroupoid if $\mathcal{K}$ is a compact open principal subgroupoid of $\mathcal{G}$ such that $\mathcal{K}^{(0)} = \mathcal{G}^{(0)}$.
- We say that $\mathcal{G}$ is an AF groupoid if it can be written as an increasing union of elementary subgroupoids.

If $\mathcal{K}$ is a compact étale principal groupoid, then $\mathcal{K}$ is identified with the equivalence relation $\{(r(g), s(g)) \mid g \in \mathcal{K}\}$ on $\mathcal{K}^{(0)}$ and the topology on $\mathcal{K}$ agrees with the relative topology from $\mathcal{K}^{(0)} \times \mathcal{K}^{(0)}$. Also, the equivalence relation $\mathcal{K}$ is uniformly finite, i.e. there exists $n \in \mathbb{N}$ such that $\# r^{-1}(x) \le n$ for any $x \in \mathcal{K}^{(0)}$.

An AF groupoid is principal by definition. The C^*-algebra associated with an AF groupoid is an AF algebra. It is known that any AF groupoids are represented by Bratteli diagrams (see [16, Theorem 3.9]). We provide a brief explanation of it. A directed graph $B = (V, E)$ is called a Bratteli diagram when $V = \bigcup_{n=0}^{\infty} V_n$ and $E = \bigcup_{n=1}^{\infty} E_n$ are disjoint unions of finite sets of vertices and edges with maps $i : E_n \to V_{n-1}$ and $t : E_n \to V_n$ both of which are surjective. Let

$$X_B = \left\{ (e_n)_n \in \prod_n E_n \mid e_n \in E_n,\ t(e_n) = i(e_{n+1}) \quad \forall n \in \mathbb{N} \right\}.$$

The set X_B endowed with the relative topology is called the infinite path space of B. Define an equivalence relation (i.e. principal groupoid) $\mathcal{K}_m$ by

$$\mathcal{K}_m = \{((e_n)_n, (f_n)_n) \in X_B \times X_B \mid e_n = f_n \quad \forall n \ge m\}.$$

Then, $\mathcal{K}_m$ equipped with the relative topology from $X_B \times X_B$ is a compact principal étale groupoid. Clearly one has $\mathcal{K}_m \subset \mathcal{K}_{m+1}$. Set $\mathcal{G} = \bigcup_m \mathcal{K}_m$. Endowed with the inductive limit topology, $\mathcal{G}$ becomes an AF groupoid. Conversely, Theorem 3.9 of [16] states that any AF groupoid arises in such a way.

Example 2.3 (Transformation Groupoids) Let $\varphi : \Gamma \curvearrowright X$ be an action of a countable discrete group Γ on a Cantor set X by homeomorphisms. We let $G_\varphi = \Gamma \times X$ and define the following groupoid structure: (γ, x) and (γ', x') are composable if and only if $x = \varphi^{\gamma'}(x')$, in which case $(\gamma, \varphi^{\gamma'}(x')) \cdot (\gamma', x') = (\gamma\gamma', x')$, and $(\gamma, x)^{-1} = (\gamma^{-1}, \varphi^{\gamma}(x))$. Then G_φ is an étale groupoid and called the transformation groupoid arising from $\varphi : \Gamma \curvearrowright X$. The unit space $\mathcal{G}_\varphi^{(0)}$ is canonically identified with X via the map $(1, x) \mapsto x$.

The groupoid $\mathcal{G}_\varphi$ is principal if and only if the action φ is free, that is, φ^γ does not have any fixed points unless $\gamma = 1$. The groupoid $\mathcal{G}_\varphi$ is essentially principal if and only if the action φ is topologically free, that is, $\{x \in X \mid \varphi^\gamma(x) = x\}$ has no interior points unless $\gamma = 1$. The groupoid $\mathcal{G}_\varphi$ is minimal if and only if the action φ is minimal, that is, any orbit of φ is dense in X.

The C^*-algebra $C_r^*(\mathscr{G}_\varphi)$ is canonically isomorphic to the crossed product C^*-algebra $C(X) \rtimes_r \Gamma$.

Medynets, Sauer and Thom recently obtained the following interesting result.

Theorem 2.4 ([31, Theorem 3.2]) *Let Γ and Λ be finitely generated groups. The following are equivalent.*

1. *There exist free actions $\varphi : \Gamma \curvearrowright X$ and $\psi : \Lambda \curvearrowright Y$ on Cantor sets such that $\mathscr{G}_\varphi \cong \mathscr{G}_\psi$.*
2. *Γ and Λ are bi-Lipschitz equivalent.*

Example 2.5 (SFT Groupoids) We recall the definition of étale groupoids arising from one-sided shifts of finite type [28, Section 6.1]. Let (V, E) be a finite directed graph, where V is a finite set of vertices and E is a finite set of edges. For $e \in E$, $i(e)$ denotes the initial vertex of e and $t(e)$ denotes the terminal vertex of e. Let $A = (A(\xi, \eta))_{\xi,\eta \in V}$ be the adjacency matrix of (V, E), that is,

$$A(\xi, \eta) = \#\{e \in E \mid i(e) = \xi,\ t(e) = \eta\}.$$

We assume that A is irreducible (i.e. for all $\xi, \eta \in V$ there exists $n \in \mathbb{N}$ such that $A^n(\xi, \eta) > 0$) and that A is not a permutation matrix. Define

$$X_A = \{(x_k)_{k\in\mathbb{N}} \in E^{\mathbb{N}} \mid t(x_k) = i(x_{k+1}) \quad \forall k \in \mathbb{N}\}.$$

With the product topology, X_A is a Cantor set. Define a surjective continuous map $\sigma_A : X_A \to X_A$ by

$$\sigma_A(x)_k = x_{k+1} \quad k \in \mathbb{N},\ x = (x_k)_k \in X_A.$$

In other words, σ_A is the (one-sided) shift on X_A. It is easy to see that σ_A is a local homeomorphism. The dynamical system (X_A, σ_A) is called the one-sided irreducible shift of finite type (SFT) associated with the graph (V, E) (or the matrix A).

The étale groupoid $\mathscr{G}_A$ for (X_A, σ_A) is given by

$$\mathscr{G}_A = \{(x, n, y) \in X_A \times \mathbb{Z} \times X_A \mid \exists k, l \in \mathbb{N},\ n = k-l,\ \sigma_A^k(x) = \sigma_A^l(y)\}.$$

The topology of $\mathscr{G}_A$ is generated by the sets $\{(x, k-l, y) \in \mathscr{G}_A \mid x \in P,\ y \in Q,\ \sigma_A^k(x) = \sigma_A^l(y)\}$, where $P, Q \subset X_A$ are open and $k, l \in \mathbb{N}$. Two elements (x, n, y) and (x', n', y') in $\mathscr{G}$ are composable if and only if $y = x'$, and the multiplication and the inverse are

$$(x, n, y) \cdot (y, n', y') = (x, n+n', y'), \quad (x, n, y)^{-1} = (y, -n, x).$$

We identify X_A with the unit space $\mathscr{G}_A^{(0)}$ via $x \mapsto (x, 0, x)$. We call $\mathscr{G}_A$ the SFT groupoid associated with the matrix A.

The groupoid $\mathscr{G}_A$ is essentially principal and minimal.

The groupoid C^*-algebra $C^*_r(\mathscr{G}_A)$ is isomorphic to the Cuntz-Krieger algebra $\mathscr{O}_A$ of [8], which is simple and purely infinite.

3 Homology Groups

The homology groups $H_n(\mathscr{G})$ of an étale groupoid $\mathscr{G}$ were first introduced and studied by Crainic and Moerdijk in [7]. In the case that the unit space $\mathscr{G}^{(0)}$ is a Cantor set, we investigated connections between the homology groups and dynamical properties of $\mathscr{G}$ in [27–29]. In this section, we would like to recall the definition of $H_n(\mathscr{G})$ for an étale groupoid $\mathscr{G}$ whose unit space is a Cantor set.

Let A be a topological abelian group. For a locally compact Hausdorff space X, we denote by $C_c(X,A)$ the set of A-valued continuous functions with compact support. When X is compact, we simply write $C(X,A)$. With pointwise addition, $C_c(X,A)$ is an abelian group. Let $\pi : X \to Y$ be a local homeomorphism between locally compact Hausdorff spaces. For $f \in C_c(X,A)$, we define a map $\pi_*(f) : Y \to A$ by

$$\pi_*(f)(y) = \sum_{\pi(x)=y} f(x).$$

It is not so hard to see that $\pi_*(f)$ belongs to $C_c(Y,A)$ and that π_* is a homomorphism from $C_c(X,A)$ to $C_c(Y,A)$. Besides, if $\pi' : Y \to Z$ is another local homeomorphism to a locally compact Hausdorff space Z, then one can check $(\pi' \circ \pi)_* = \pi'_* \circ \pi_*$ in a direct way. Thus, $C_c(\cdot,A)$ is a covariant functor from the category of locally compact Hausdorff spaces with local homeomorphisms to the category of abelian groups with homomorphisms.

Let $\mathscr{G}$ be an étale groupoid. For $n \in \mathbb{N}$, we write $\mathscr{G}^{(n)}$ for the space of composable strings of n elements in $\mathscr{G}$, that is,

$$\mathscr{G}^{(n)} = \{(g_1, g_2, \dots, g_n) \in \mathscr{G}^n \mid s(g_i) = r(g_{i+1}) \text{ for all } i = 1, 2, \dots, n-1\}.$$

For $i = 0, 1, \dots, n$, we let $d_i : G^{(n)} \to G^{(n-1)}$ be a map defined by

$$d_i(g_1, g_2, \dots, g_n) = \begin{cases} (g_2, g_3, \dots, g_n) & i = 0 \\ (g_1, \dots, g_i g_{i+1}, \dots, g_n) & 1 \le i \le n-1 \\ (g_1, g_2, \dots, g_{n-1}) & i = n. \end{cases}$$

When $n = 1$, we let $d_0, d_1 : \mathscr{G}^{(1)} \to \mathscr{G}^{(0)}$ be the source map and the range map, respectively. Clearly the maps d_i are local homeomorphisms.

Define the homomorphisms $\partial_n : C_c(\mathscr{G}^{(n)}, A) \to C_c(\mathscr{G}^{(n-1)}, A)$ by

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_{i*}.$$

It is easy see that the abelian groups $C_c(\mathscr{G}^{(n)}, A)$ together with the boundary operators ∂_n form a chain complex.

Definition 3.1 ([7, Section 3.1], [27, Definition 3.1]) We let $H_n(\mathscr{G}, A)$ be the homology groups of the Moore complex above, i.e. $H_n(\mathscr{G}, A) = \ker \partial_n / \operatorname{Im} \partial_{n+1}$, and call them the homology groups of $\mathscr{G}$ with constant coefficients A. When $A = \mathbb{Z}$, we simply write $H_n(\mathscr{G}) = H_n(\mathscr{G}, \mathbb{Z})$. In addition, we define

$$H_0(\mathscr{G})^+ = \{[f] \in H_0(\mathscr{G}) \mid f(x) \geq 0 \text{ for all } x \in \mathscr{G}^{(0)}\},$$

where $[f]$ denotes the equivalence class of $f \in C_c(\mathscr{G}^{(0)}, \mathbb{Z})$.

Remark 3.2 The pair $(H_0(\mathscr{G}), H_0(\mathscr{G})^+)$ is not necessarily an ordered abelian group in general, because $H_0(\mathscr{G})^+ \cap (-H_0(\mathscr{G})^+)$ may not equal $\{0\}$. In fact, when $\mathscr{G}$ is the SFT groupoid, $H_0(G)^+ = H_0(G)$.

Example 3.3

1. Let $\mathscr{G}$ be an AF groupoid (see Example 2.2). There exists an isomorphism $\pi : H_0(\mathscr{G}) \to K_0(C_r^*(\mathscr{G}))$ such that $\pi(H_0(\mathscr{G})^+) = K_0(C_r^*(\mathscr{G}))^+$ and $\pi([1_{\mathscr{G}^{(0)}}]) = [1]$ [27, Theorem 4.10]. For $n \geq 1$, we have $H_n(\mathscr{G}) = 0$ [27, Theorem 4.11].
2. Let $\mathscr{G}_\varphi$ be the transformation groupoid associated with a group action $\varphi : \Gamma \curvearrowright X$ (see Example 2.3). Then the homology groups $H_n(\mathscr{G})$ is naturally isomorphic to the usual group homology $H_n(\Gamma, C(X, \mathbb{Z}))$ of Γ with coefficients in $C_c(X, \mathbb{Z})$ (see [5, Chapter III]).
3. Let $\mathscr{G}_A$ be an SFT groupoid, where A is the adjacency matrix of an irreducible finite directed graph (V, E) (see Example 2.5). The matrix A acts on the abelian group $\mathbb{Z}^V$ by multiplication. Then one has

$$H_n(\mathscr{G}_A) \cong \begin{cases} \operatorname{Coker}(\mathrm{id} - A^t) & n = 0 \\ \operatorname{Ker}(\mathrm{id} - A^t) & n = 1 \\ 0 & n \geq 2. \end{cases}$$

   See [27, Theorem 4.14].

The cohomology groups $H^n(\mathscr{G})$ of an étale groupoid $\mathscr{G}$ were introduced by Renault in [35] and have been studied by many authors. When $\mathscr{G}_\varphi$ is the transformation groupoid associated with a group action $\varphi : \Gamma \curvearrowright X$, the cohomology $H^n(\mathscr{G}_\varphi)$ is canonically isomorphic to the usual group cohomology $H^n(\Gamma, C(X, \mathbb{Z}))$. In particular, when $\Gamma = \mathbb{Z}^N$, there exist natural isomorphisms $H_n(\mathscr{G}_\varphi) \cong H^{N-n}(\mathscr{G}_\varphi)$

(Poincaré duality). In general, however, we do not know if any connections exist between $H_*(\mathscr{G})$ and $H^*(\mathscr{G})$.

For the homology groups $H_n(\mathscr{G})$, the following Künneth theorem holds.

Theorem 3.4 ([29, Theorem 2.4]) *Let $\mathscr{G}$ and $\mathscr{H}$ be étale groupoids. For any $n \geq 0$, there exists a natural short exact sequence*

$$0 \longrightarrow \bigoplus_{i+j=n} H_i(\mathscr{G}) \otimes H_j(\mathscr{H})$$

$$\longrightarrow H_n(\mathscr{G} \times \mathscr{H}) \longrightarrow \bigoplus_{i+j=n-1} \mathrm{Tor}(H_i(\mathscr{G}), H_j(\mathscr{H})) \longrightarrow 0.$$

Furthermore these sequences split (but not canonically).

In [29, Section 2.3], we made the following conjecture about homology groups $H_n(\mathscr{G})$ and K-groups $K_i(C_r^*(\mathscr{G}))$.

Conjecture 3.5 (HK Conjecture) Let $\mathscr{G}$ be an essentially principal minimal étale groupoid whose unit space $\mathscr{G}^{(0)}$ is a Cantor set. Then we have

$$\bigoplus_{i=0}^{\infty} H_{2i}(\mathscr{G}) \cong K_0(C_r^*(\mathscr{G}))$$

and

$$\bigoplus_{i=0}^{\infty} H_{2i+1}(\mathscr{G}) \cong K_1(C_r^*(\mathscr{G})).$$

Example 3.6 The HK conjecture is true for any AF groupoid $\mathscr{G}$. This is clear from Example 3.3 (1).

Example 3.7 Let $\mathscr{G}_\varphi$ be the transformation groupoid associated with a group action $\varphi : \Gamma \curvearrowright X$. In general, it is not known whether the HK conjecture holds for $\mathscr{G}_\varphi$. When $\Gamma = \mathbb{Z}$, the Pimsner-Voiculescu exact sequence implies that the HK conjecture is true. Suppose $\Gamma = \mathbb{Z}^N$. The suspension space Ω_φ is the quotient space of $\mathbb{R}^N \times X$ by the equivalence relation

$$(t,x) \sim (t',x') \iff t - t' \in \mathbb{Z}^N,\ \varphi^{t-t'}(x) = x'.$$

The translation $(t,x) \mapsto (t+s,x)$ gives rise to an action $\tilde{\varphi} : \mathbb{R}^N \curvearrowright \Omega_\varphi$. It is well-known that $C(\Omega_\varphi) \rtimes_{\tilde{\varphi}} \mathbb{R}^N$ is stably isomorphic to $C(X) \rtimes_\varphi \mathbb{Z}^N$. Then we have

$$\begin{aligned} K_i(C_r^*(\mathscr{G}_\varphi)) &= K_i(C(X) \rtimes_\varphi \mathbb{Z}^N) \cong K_i(C(\Omega_\varphi) \rtimes_{\tilde{\varphi}} \mathbb{R}^N) \\ &\cong K_{i+N}(C(\Omega_\varphi)) \qquad \because \text{Thom Isomorphism} \\ &= K^{i+N}(\Omega_\varphi). \end{aligned}$$

On the other hand, we know $H_n(\mathscr{G}_\varphi) = H_n(\mathbb{Z}^N, C(X,\mathbb{Z}))$ is naturally isomorphic to $\check{H}^n(\Omega_\varphi, \mathbb{Z})$, where $\check{H}$ denotes the Čech cohomology. (This is a folklore fact and I don't know an appropriate reference. A relevant remark can be found in the final paragraph of [5, Chapter III.1].) By the Chern character, there exist isomorphisms $K^*(\Omega_\varphi) \otimes \mathbb{Q} \to \bigoplus H^*(\Omega_\varphi, \mathbb{Z}) \otimes \mathbb{Q}$. It follows that there exist isomorphisms

$$\bigoplus_{i=0}^{\infty} H_{2i}(\mathscr{G}_\varphi) \otimes \mathbb{Q} \cong K_0(C_r^*(\mathscr{G}_\varphi)) \otimes \mathbb{Q}$$

and

$$\bigoplus_{i=0}^{\infty} H_{2i+1}(\mathscr{G}_\varphi) \otimes \mathbb{Q} \cong K_1(C_r^*(\mathscr{G}_\varphi)) \otimes \mathbb{Q}.$$

The HK conjecture asks if the integral version of these isomorphisms is true or not.

For more general group actions $\Gamma \curvearrowright X$, the HK conjecture is wide open.

Example 3.8 Let $\mathscr{G}_A$ be an SFT groupoid, where A is the adjacency matrix of an irreducible finite directed graph (V, E). By Example 3.3 (3), we can see $H_i(\mathscr{G}_A) \cong K_i(C_r^*(\mathscr{G}_A))$ for $i = 1, 2$. Therefore, the HK conjecture holds for $\mathscr{G}_A$.

4 Topological Full Groups

In this section, we introduce the definition of topological full groups.

Definition 4.1 ([27, Definition 2.3]) Let $\mathscr{G}$ be an essentially principal étale groupoid whose unit space $\mathscr{G}^{(0)}$ is a Cantor set. The set of all $\alpha \in \mathrm{Homeo}(\mathscr{G}^{(0)})$ for which there exists a compact open $\mathscr{G}$-set U satisfying $\alpha = \theta(U)$ is called the topological full group of $\mathscr{G}$ and denoted by $[[\mathscr{G}]]$.

For $\alpha \in [[\mathscr{G}]]$ the compact open $\mathscr{G}$-set U as above uniquely exists, because $\mathscr{G}$ is essentially principal. Obviously $[[\mathscr{G}]]$ is a subgroup of $\mathrm{Homeo}(\mathscr{G}^{(0)})$. Since $\mathscr{G}$ is second countable, it has countably many compact open subsets, and so $[[\mathscr{G}]]$ is at most countable.

A homeomorphism $\alpha : \mathscr{G}^{(0)} \to \mathscr{G}^{(0)}$ belongs to $[[\mathscr{G}]]$ if and only if for any $x \in \mathscr{G}^{(0)}$ there exists a compact open $\mathscr{G}$-set V such that x is in $s(V)$ and α equals $\theta(V)$ on a neighborhood of x. Thus, α is in $[[\mathscr{G}]]$ if and only if the 'graph' of α is a clopen subset of $\mathscr{G}$.

Example 4.2 Let $\mathscr{G}$ be an AF groupoid arising from a Bratteli diagram (V, E) (see Example 2.2). For each $k \in \mathbb{N}$ and $v \in V_k$, we let E_v be the set of paths from a vertex in V_0 to the vertex v, i.e.

$$E_v = \{(e_1, e_2, \ldots, e_k) \mid t(e_n) = i(e_{n+1}),\ t(e_k) = v\}.$$

Suppose that a permutation σ_v on E_v is given. Then we can define $\tilde{\sigma}_v \in [[\mathscr{G}]]$ by

$$\tilde{\sigma}_v((e_n)_n) = \begin{cases} (\sigma_v(e_1, e_2, \dots, e_k), e_{k+1}, \dots) & t(e_k) = v \\ (e_n)_n & t(e_k) \neq v. \end{cases}$$

Let $G_k \subset [[\mathscr{G}]]$ be a subgroup generated by $\tilde{\sigma}_v$ for vertices $v \in V_k$ and permutations σ_v. It is easy to see

$$G_k \cong \bigoplus_{v \in V_k} \mathfrak{S}_{\#E_v}, \quad G_k \subset G_{k+1}$$

and $[[\mathscr{G}]] = \bigcup_k G_k$. Thus, $[[\mathscr{G}]]$ is an increasing union of subgroups isomorphic to finite direct sums of symmetric groups. Conversely, it is known that if $[[\mathscr{G}]]$ is locally finite, then $\mathscr{G}$ is AF [26, Proposition 3.2].

Example 4.3 Let $\mathscr{G}_\varphi$ be the transformation groupoid associated with a group action $\varphi : \Gamma \curvearrowright X$ (see Example 2.3). Suppose that $\mathscr{G}_\varphi$ is essentially principal. Take $\alpha \in [[\mathscr{G}_\varphi]]$. There exists a compact open $\mathscr{G}_\varphi$-set U such that $\alpha = \theta(U)$. We can find a continuous map $c : X \to \Gamma$ such that $U = \{(c(x), x) \mid x \in X\}$. It follows that $\alpha(x) = \varphi^{c(x)}(x)$ for all $x \in X$.

Example 4.4 Let $\mathscr{G}_A$ be an SFT groupoid, where A is the adjacency matrix of an irreducible finite directed graph (V, E) (see Example 2.5). We say that $\mu = (e_1, e_2, \dots, e_k) \in E^k$ is a path if $t(e_j) = i(e_{j+1})$ for every $j = 1, 2, \dots, k-1$. The terminal vertex of μ is written $t(\mu) = t(e_k)$. The length of μ is written $|\mu| = k$. For a path $\mu = (e_1, e_2, \dots, e_k)$,

$$C_\mu = \{(x_n)_{n\in\mathbb{N}} \in X_A \mid x_j = e_j \quad \forall j = 1, 2, \dots, k\}$$

is a clopen subset of X_A and is called a cylinder set. For two paths μ and ν with $t(\mu) = t(\nu)$, we define a compact open $\mathscr{G}_A$-set $U_{\mu,\nu}$ by

$$U_{\mu,\nu} = \{(x, |\mu|-|\nu|, y) \in \mathscr{G}_A \mid \sigma_A^{|\mu|}(x) = \sigma_A^{|\nu|}(y),\ x \in C_\mu,\ y \in C_\nu\}.$$

The subsets $U_{\mu,\nu}$ form a base for the topology of $\mathscr{G}_A$.

Take $\alpha \in [[\mathscr{G}_A]]$. There exist $\mu_1, \mu_2, \dots, \mu_m$ and $\nu_1, \nu_2, \dots, \nu_m$ such that the following hold.

- $\{C_{\mu_i} \mid i = 1, 2, \dots, m\}$ is a clopen partition of X_A.
- $\{C_{\nu_i} \mid i = 1, 2, \dots, m\}$ is a clopen partition of X_A.
- $t(\mu_i) = t(\nu_i)$ for every i.
- $U = \bigcup_i U_{\mu_i,\nu_i}$ is a compact open $\mathscr{G}$-set satisfying $\alpha = \theta(U)$.

Let us consider the simplest case, namely that A is a 1×1 matrix $[n]$. The groupoid C^*-algebra $C_r^*(\mathscr{G}_{[n]})$ is the Cuntz algebra $\mathcal{O}_n$. Nekrashevych [32, Proposition 9.6] observed that the topological full group $[[\mathscr{G}_{[n]}]]$ is naturally isomorphic to the

Higman-Thompson group $V_{n,1}$. The group $V_{n,r}$ is defined to be the group of all right continuous PL bijections $f : [0,r) \to [0,r)$ with finitely many singularities such that all singularities of f are in $\mathbb{Z}[1/n]$, the derivative of f at any non-singular point is n^k for some $k \in \mathbb{Z}$ and f maps $\mathbb{Z}[1/n] \cap [0,r)$ to itself. The isomorphism $\pi : [[\mathscr{G}_{[n]}]] \to V_{n,1}$ is described as follows. We identify the shift space $X_{[n]}$ with $\{0,1,\dots,n-1\}^{\mathbb{N}}$. Define a continuous map $F : X_{[n]} \to [0,1]$ by

$$F((x_k)_k) = \sum_{k=1}^{\infty} \frac{x_k}{n^k}.$$

Then we have $\pi(\alpha) \circ F = F \circ \alpha$ for any $\alpha \in [[\mathscr{G}_{[n]}]]$.

For an SFT groupoid $\mathscr{G}_A$, $[[\mathscr{G}_A]]$ is thought of as a generalization of the Higman-Thompson group $V_{n,1}$.

When $\alpha = \theta(U)$ is an element of $[[\mathscr{G}]]$, $1_U \in C_c(\mathscr{G})$ can be thought of as a unitary in $C_r^*(\mathscr{G})$. This unitary 1_U normalizes $C(\mathscr{G}^{(0)})$, namely $1_U f 1_U^* = f \circ \alpha$ holds for every $f \in C(\mathscr{G}^{(0)})$. Conversely, if $u \in C_r^*(\mathscr{G})$ is a normalizer of $C(\mathscr{G}^{(0)})$ and $ufu^* = f \circ \alpha$, then α belongs to $[[\mathscr{G}]]$. Thus, we have the following.

Proposition 4.5 ([27, Proposition 5.6]) *Suppose that $\mathscr{G}$ is an essentially principal étale groupoid whose unit space $\mathscr{G}^{(0)}$ is a Cantor set. There exists a natural short exact sequence*

$$1 \longrightarrow U(C(\mathscr{G}^{(0)})) \longrightarrow N(C(\mathscr{G}^{(0)}), C_r^*(\mathscr{G})) \xrightarrow{\ \sigma\ } [[\mathscr{G}]] \longrightarrow 1,$$

where $N(C(\mathscr{G}^{(0)}), C_r^*(\mathscr{G}))$ denotes the group of unitary normalizers of $C(\mathscr{G}^{(0)})$ in $C_r^*(\mathscr{G})$. Furthermore, the homomorphism σ has a right inverse.

Next, we would like to introduce the index map $I : [[\mathscr{G}]] \to H_1(\mathscr{G})$.

Definition 4.6 ([27, Definition 7.1]) Let $\mathscr{G}$ be an essentially principal étale groupoid whose unit space is a Cantor set. For $\alpha \in [[\mathscr{G}]]$, a compact open $\mathscr{G}$-set U satisfying $\alpha = \theta(U)$ uniquely exists. It is easy to see that 1_U is a 1-cycle, i.e. $1_U \in \operatorname{Ker} \partial_1$ (see Sect. 3). We define a map $I : [[\mathscr{G}]] \to H_1(G)$ by $I(\alpha) = [1_U]$ and call it the index map.

It is easy to check that $I : [[\mathscr{G}]] \to H_1(\mathscr{G})$ is a homomorphism. We write $[[\mathscr{G}]]_0 = \operatorname{Ker} I$. For $\alpha = \theta(U) \in [[\mathscr{G}]]$, the element 1_U may be regarded as a unitary of $C_r^*(\mathscr{G})$, and so we can think about its K_1-class $[1_U] \in K_1(C_r^*(\mathscr{G}))$. It is a natural open question to find a connection between $[1_U] \in K_1(C_r^*(\mathscr{G}))$ and $I(\alpha) \in H_1(\mathscr{G})$.

We made the following conjecture about abelianization $[[\mathscr{G}]]_{\mathrm{ab}}$ and homology groups $H_n(\mathscr{G})$ in [29, Section 2.3].

Conjecture 4.7 (AH conjecture) Let $\mathscr{G}$ be an essentially principal minimal étale groupoid whose unit space $\mathscr{G}^{(0)}$ is a Cantor set. Then there exists an exact sequence

$$H_0(\mathscr{G}) \otimes \mathbb{Z}_2 \longrightarrow [[\mathscr{G}]]_{\mathrm{ab}} \xrightarrow{I} H_1(\mathscr{G}) \longrightarrow 0.$$

Especially, if $H_0(\mathscr{G})$ is 2-divisible, then we have $[[\mathscr{G}]]_{\mathrm{ab}} \cong H_1(\mathscr{G})$.

Indeed, in many examples we can verify that there exists a short exact sequence

$$0 \longrightarrow H_0(\mathscr{G}) \otimes \mathbb{Z}_2 \xrightarrow{j} [[\mathscr{G}]]_{\mathrm{ab}} \xrightarrow{I} H_1(\mathscr{G}) \longrightarrow 0.$$

In such a case, we say that $\mathscr{G}$ satisfies the strong AH property. There exists $\mathscr{G}$ which does not satisfy the strong AH property (but the AH conjecture is still true). See [33, Example 7.1], [29, Section 5.5] or Theorem 8.12 (4).

The homomorphism $H_0(\mathscr{G}) \otimes \mathbb{Z}_2 \to [[\mathscr{G}]]_{\mathrm{ab}}$ appearing in the AH conjecture is described as follows. Let U be a compact open $\mathscr{G}$-set satisfying $r(U) \cap s(U) = \emptyset$. Define $\tau \in [[\mathscr{G}]]$ by

$$\tau(x) = \begin{cases} \theta(U)(x) & x \in s(U) \\ \theta(U^{-1})(x) & x \in r(U) \\ x & \text{otherwise.} \end{cases}$$

The homomorphism $H_0(\mathscr{G}) \otimes \mathbb{Z}_2 \to [[\mathscr{G}]]_{\mathrm{ab}}$ sends the equivalence class of $1_{s(U)} \otimes \bar{1}$ to the equivalence class of τ. Notice that it is not clear at all if this is really well-defined. One can find a proof of the well-definedness in [33].

Example 4.8

1. Let $\mathscr{G}$ be an AF groupoid. In Example 4.2, we have seen that $[[\mathscr{G}]]$ is an increasing union of the subgroups $G_k \cong \bigoplus_{v \in V_k} \mathfrak{S}_{\#E_v}$. Clearly $(G_k)_{\mathrm{ab}}$ is isomorphic to $\bigoplus_{v \in V_k} \mathbb{Z}_2$. So, $[[\mathscr{G}]]_{\mathrm{ab}}$ is an inductive limit of $\bigoplus_{v \in V_k} \mathbb{Z}_2$, and the connecting maps are given by the edge sets E_k. Hence $[[\mathscr{G}]]_{\mathrm{ab}}$ is isomorphic to $H_0(\mathscr{G}) \otimes \mathbb{Z}_2$ (see [26, Section 3]). In particular, the AF groupoid $\mathscr{G}$ has the strong AH property.
2. Let $\mathscr{G}_\varphi$ be the transformation groupoid associated with a minimal group action $\varphi : \Gamma \curvearrowright X$. When Γ is $\mathbb{Z}$, it was shown that $[[\mathscr{G}_\varphi]]_{\mathrm{ab}}$ is isomorphic to $H_1(\mathscr{G}_\varphi) \oplus (H_0(\mathscr{G}_\varphi) \otimes \mathbb{Z}_2)$ [26, Section 4]. Thus, $\mathscr{G}_\varphi$ has the strong AH property. When Γ is $\mathbb{Z}^N$, we can prove that the AH conjecture holds for $\mathscr{G}_\varphi$ (see Theorem 6.5 (3)). But, we do not know whether or not $\mathscr{G}_\varphi$ satisfies the strong AH property. For other group actions $\Gamma \curvearrowright X$, nothing is known.
3. Let $\mathscr{G}_A$ be an SFT groupoid. It was proved that the abelianization $[[\mathscr{G}_A]]_{\mathrm{ab}}$ is isomorphic to $H_1(\mathscr{G}_A) \oplus (H_0(\mathscr{G}_A) \otimes \mathbb{Z}_2)$ [28, Corollary 6.24]. Thus, $\mathscr{G}_A$ has the strong AH property. See Theorem 8.6.

5 Isomorphism Theorem

In this section, we discuss the following theorem.

Theorem 5.1 ([28, Theorem 3.10]) *For $i = 1, 2$, let $\mathscr{G}_i$ be an essentially principal étale groupoid whose unit space is a Cantor set. Suppose that $\mathscr{G}_i$ is minimal. The following conditions are equivalent.*

1. *$\mathscr{G}_1$ and $\mathscr{G}_2$ are isomorphic as étale groupoids.*
2. *$[[\mathscr{G}_1]]$ and $[[\mathscr{G}_2]]$ are isomorphic as discrete groups.*
3. *$[[\mathscr{G}_1]]_0$ and $[[\mathscr{G}_2]]_0$ are isomorphic as discrete groups.*
4. *$D([[\mathscr{G}_1]])$ and $D([[\mathscr{G}_2]])$ are isomorphic as discrete groups.*

It clear that condition (1) implies the other conditions. The reverse implications are nontrivial, and the essential part of the proof is contained in the following proposition.

Proposition 5.2 ([28, Theorem 3.5, Proposition 3.6]) *For $i = 1, 2$, let $\mathscr{G}_i$ be an essentially principal étale groupoid whose unit space is a Cantor set. Suppose that $\mathscr{G}_i$ is minimal. For each $i = 1, 2$, let Γ_i be a subgroup of $[[\mathscr{G}_i]]$ such that $D([[\mathscr{G}_i]]) \subset \Gamma_i$. If there exists an isomorphism $\varphi : \Gamma_1 \to \Gamma_2$, then there exists a homeomorphism $\varphi : \mathscr{G}_1^{(0)} \to \mathscr{G}_2^{(0)}$ such that $\varphi(\alpha) = \varphi \circ \alpha \circ \varphi^{-1}$ for all $\alpha \in \Gamma_1$.*

This proposition says that any isomorphism between the groups Γ_1 and Γ_2 is spatially realized by a homeomorphism between the unit spaces. If we get a homeomorphism $\varphi : \mathscr{G}_1^{(0)} \to \mathscr{G}_2^{(0)}$ such that $\varphi(\alpha) = \varphi \circ \alpha \circ \varphi^{-1}$, then it is not so hard to see that φ gives rise to an isomorphism from $\mathscr{G}_1$ to $\mathscr{G}_2$. Therefore we can prove the remaining implications of Theorem 5.1.

In the setting of topological dynamical systems, Theorem 5.1 was first proved by Giordano, Putnam and Skau [15] for minimal $\mathbb{Z}$-actions. In order to obtain the spatial realization (see the proposition above), they imported the method of H. Dye, who proved the same isomorphism result for measure preserving ergodic actions on Lebesgue spaces. We remark that for minimal $\mathbb{Z}$-actions φ_1 and φ_2, $\mathscr{G}_{\varphi_1}$ is isomorphic to $\mathscr{G}_{\varphi_2}$ if and only if φ_1 is flip conjugate to φ_2 [14, Theorem 2.4]. See Theorem 8.1. Later, similar results were obtained by Bezuglyi and Medynets [1] and by Medynets [30]. The proof of Theorem 5.1 given in [28] is along the same line as these works.

The proposition above can be also thought of as an immediate consequence of the following theorem of Rubin [37] (see also [3, Section 9], [33, Section 3.3]). Let X be a topological space. We say that a subgroup $\Gamma \subset \mathrm{Homeo}(X)$ is locally dense if for every $x \in X$ and every open set $U \subset X$ with $x \in U$, the closure of

$$\{f(x) \mid f \in \Gamma, f|(X \setminus U) = \mathrm{id}\,|(X \setminus U)\}$$

has nonempty interior.

Theorem 5.3 ([37, Corollary 3.5]) *For $i = 1, 2$, let X_i be a locally compact, Hausdorff topological spaces without isolated points and let $\Gamma_i \subset \mathrm{Homeo}(X_i)$ be subgroups. If Γ_1 and Γ_2 are isomorphic and are both locally dense, then for any isomorphism $\varphi : \Gamma_1 \to \Gamma_2$ there exists a unique homeomorphism $\varphi : X_1 \to X_2$ such that $\varphi(\alpha) = \varphi \circ \alpha \circ \varphi^{-1}$ for all $\alpha \in \Gamma_1$.*

Recently, Nekrashevych [33] introduced two normal subgroups $\mathrm{A}(\mathscr{G}) \subset \mathrm{S}(\mathscr{G}) \subset [[\mathscr{G}]]$. Roughly speaking, $\mathrm{S}(\mathscr{G})$ is the subgroup generated by all elements of order two, and $\mathrm{A}(\mathscr{G})$ is the subgroup generated by all elements of order three (see [33] for the precise definitions). They are analogs of the symmetric and alternating groups. He proved that the same statement as Theorem 5.1 is true for $\mathrm{A}(\mathscr{G})$ and $\mathrm{S}(\mathscr{G})$.

6 Almost Finite Groupoids

In this section, we list known and unknown properties of almost finite groupoids. Let us begin with the definition.

Definition 6.1 ([27, Definition 6.2]) Let $\mathscr{G}$ be an essentially principal étale groupoid whose unit space is a Cantor set. We say that $\mathscr{G}$ is almost finite if for any compact subset $C \subset \mathscr{G}$ and $\varepsilon > 0$ there exists an elementary subgroupoid $\mathscr{K} \subset \mathscr{G}$ such that

$$\frac{\#(C\mathscr{K}x \setminus \mathscr{K}x)}{\#(\mathscr{K}x)} < \varepsilon$$

for all $x \in \mathscr{G}^{(0)}$. We also remark that $\#(\mathscr{K}(x))$ equals $\#(\mathscr{K}x)$, because $\mathscr{K}$ is principal.

I remark that the idea of the definition above has its origin in the work of Latrémolière and Ormes [22]. More precisely, the notion of almost finiteness was made so that the arguments of [22] proceed in an analogous way.

AF groupoids (see Example 2.2) are almost finite. Indeed, any compact subset $C \subset \mathscr{G}$ is contained in an elementary subgroupoid $\mathscr{K} \subset \mathscr{G}$. As the next proposition shows, there exist almost finite groupoids which are not AF.

Proposition 6.2 ([27, Lemma 6.3]) *When $\varphi : \mathbb{Z}^N \curvearrowright X$ is a free action of $\mathbb{Z}^N$ on a Cantor set X, the transformation groupoid $\mathscr{G}_\varphi$ is almost finite.*

Definition 6.1 may remind the reader of the Følner condition for amenable groups. It may be natural to expect that transformation groupoids arising from free actions of amenable groups are almost finite. This is an important open question. In fact, X. Li recently proved that the converse is true.

Proposition 6.3 ([23]) *Let $\varphi : \Gamma \curvearrowright X$ be a topologically free action of a discrete countable group Γ on a Cantor set X. If $\mathscr{G}_\varphi$ is almost finite, then Γ is amenable.*

It is also an interesting problem to compare the notion of almost finite groupoids with almost AF groupoids, which were introduced by Phillips [34].

For a $\mathcal{G}$-invariant probability measure $\mu \in M(\mathcal{G})$, we can define a homomorphism $\hat{\mu} : H_0(\mathcal{G}) \to \mathbb{R}$ by

$$\hat{\mu}([f]) = \int f \, d\mu.$$

It is clear that $\hat{\mu}([1_{\mathcal{G}^{(0)}}]) = 1$ and $\hat{\mu}(H_0(\mathcal{G})^+) \subset [0, \infty)$. Thus $\hat{\mu}$ is a state on $(H_0(\mathcal{G}), H_0(\mathcal{G})^+, [1_{\mathcal{G}^{(0)}}])$. It is also easy to see that the map $\mu \mapsto \hat{\mu}$ gives an isomorphism from $M(\mathcal{G})$ to the state space.

Theorem 6.4 ([29, Theorem 3.4]) *Let $\mathcal{G}$ be a minimal almost finite groupoid.*

1. *$(H_0(\mathcal{G}), H_0(\mathcal{G})^+)$ is a simple, weakly unperforated, ordered abelian group with the Riesz interpolation property.*
2. *The homomorphism $\rho : H_0(\mathcal{G}) \to \mathrm{Aff}(M(\mathcal{G}))$ defined by $\rho([f])(\mu) = \hat{\mu}([f])$ has uniformly dense range, where $\mathrm{Aff}(M(\mathcal{G}))$ denotes the space of $\mathbb{R}$-valued affine continuous functions on $M(\mathcal{G})$.*

For topological full groups of almost finite groupoids, the following are known.

Theorem 6.5 ([27–29]) *Let $\mathcal{G}$ be an almost finite groupoid.*

1. *If $\mathcal{G}$ is minimal, then the commutator subgroup $D([[\mathcal{G}]])$ is simple.*
2. *The index map $I : [[G]] \to H_1(\mathcal{G})$ is surjective.*
3. *If $\mathcal{G}$ is principal and minimal, then the AH conjecture holds for $\mathcal{G}$.*

Nekrashevych [33] recently proved that if an almost finite groupoid $\mathcal{G}$ is minimal and expansive, then $D([[\mathcal{G}]])$ is finitely generated (see [33] for the definition of expansive groupoids).

In general, it is not known if every almost finite groupoid $\mathcal{G}$ satisfies the strong AH property or not. In other words, we cannot prove that the homomorphism $j : H_0(\mathcal{G}) \otimes \mathbb{Z}_2 \to [[\mathcal{G}]]_{\mathrm{ab}}$ is always injective, and cannot find an example such that the map j has nontrivial kernel. However, for a minimal free action $\varphi : \mathbb{Z}^N \curvearrowright X$, one can prove that the kernel of j is at least 'contained' in the infinitesimal subgroup of $H_0(\mathcal{G}_\varphi)$, which is defined by

$$\mathrm{Inf}(H_0(\mathcal{G}_\varphi)) = \left\{ [f] \in H_0(\mathcal{G}_\varphi) \mid \int f \, d\mu = 0 \quad \forall \mu \in M(\mathcal{G}_\varphi) \right\}.$$

Proposition 6.6 ([29, Proposition 3.7]) *Let $\varphi : \mathbb{Z}^N \curvearrowright X$ be a minimal free action of $\mathbb{Z}^N$ on a Cantor set X. The kernel of the homomorphism $j : H_0(\mathcal{G}_\varphi) \otimes \mathbb{Z}_2 \to [[\mathcal{G}_\varphi]]_{\mathrm{ab}}$ is contained in $\mathrm{Inf}(H_0(\mathcal{G}_\varphi)) \otimes \mathbb{Z}_2$. In particular, when $\mathrm{Inf}(H_0(\mathcal{G}_\varphi)) \otimes \mathbb{Z}_2$ is trivial, $\mathcal{G}_\varphi$ has the strong AH property.*

Let $\varphi : \Gamma \curvearrowright X$ be a free minimal action of an amenable group Γ on a Cantor set X. It is known that the topological full group $[[\mathscr{G}_\varphi]]$ is sofic ([10, Proposition 5.1 (1)], see also [11, Section 3]). For $\Gamma = \mathbb{Z}$, we discuss the amenability of $[[\mathscr{G}_\varphi]]$ in Sect. 8.1.

7 Purely Infinite Groupoids

In this section, we list known and unknown properties of purely infinite groupoids. Let us begin with the definition.

Definition 7.1 ([28, Definition 4.9]) Let $\mathscr{G}$ be an essentially principal étale groupoid whose unit space is a Cantor set.

1. A clopen set $A \subset \mathscr{G}^{(0)}$ is said to be properly infinite if there exist compact open $\mathscr{G}$-sets $U, V \subset \mathscr{G}$ such that $s(U) = s(V) = A$, $r(U) \cup r(V) \subset A$ and $r(U) \cap r(V) = \emptyset$.
2. We say that $\mathscr{G}$ is purely infinite if every clopen set $A \subset \mathscr{G}^{(0)}$ is properly infinite.

If $\mathscr{G}^{(0)}$ is properly infinite, then the space $M(\mathscr{G})$ of $\mathscr{G}$-invariant probability measure on $\mathscr{G}^{(0)}$ is empty. In addition, $[[\mathscr{G}]]$ contains the free product $\mathbb{Z}_2 * \mathbb{Z}_3$ [28, Proposition 4.10], and hence $[[\mathscr{G}]]$ is not amenable. If $\mathscr{G}$ is purely infinite, then for any nonempty clopen set $Y \subset \mathscr{G}^{(0)}$, the reduction $\mathscr{G}|Y$ is again purely infinite. The SFT groupoids $\mathscr{G}_A$ (see Example 2.5) are typical examples of purely infinite minimal groupoids. When $\mathscr{G}$ is purely infinite and minimal, the reduced groupoid C^*-algebra $C^*_r(\mathscr{G})$ is purely infinite and simple.

Compare the following theorem with Theorem 6.4

Theorem 7.2 ([28, Lemma 5.3]) *Let $\mathscr{G}$ be a purely infinite groupoid. For any $h \in H_0(\mathscr{G})$, there exists a non-empty clopen set $A \subset \mathscr{G}^{(0)}$ such that $[1_A] = h$. In particular, $H_0(\mathscr{G}) = H_0(\mathscr{G})^+$.*

For topological full groups of purely infinite groupoids, the following are known.

Theorem 7.3 ([28]) *Let $\mathscr{G}$ be a purely infinite groupoid.*

1. *If $\mathscr{G}$ is minimal, then the commutator subgroup $D([[\mathscr{G}]])$ is simple.*
2. *The index map $I : [[G]] \to H_1(\mathscr{G})$ is surjective.*

Nekrashevych [33] recently proved that if a purely infinite groupoid $\mathscr{G}$ is minimal and expansive, then $D([[\mathscr{G}]])$ is finitely generated (see [33] for the definition of expansive groupoids).

It is not known if all minimal purely infinite groupoids satisfy the AH conjecture. Here, we describe a situation where the AH is inherited from a smaller groupoid to a larger groupoid.

Proposition 7.4 ([29, Theorem 4.4, Proposition 4.5]) *Let $\mathscr{G}$ be a minimal étale groupoid. Let $c : \mathscr{G} \to \mathbb{Z}$ be a continuous surjective homomorphism and let $\mathscr{H} = \operatorname{Ker} c$. Assume either of the following conditions.*

1. *$\mathscr{H}$ is a principal, minimal, almost finite groupoid with $M(\mathscr{H}) = \{\mu\}$, and there exists a real number $0 < \lambda < 1$ such that, for any compact open $\mathscr{G}$-set $U \subset c^{-1}(1)$, $\mu(r(U)) = \lambda\mu(s(U))$ holds.*
2. *$\mathscr{H}$ is a minimal, purely infinite groupoid satisfying the AH conjecture.*

Then, $\mathscr{G}$ is purely infinite and satisfies the AH conjecture.

Example 7.5 Let $\mathscr{G}_A$ be an SFT groupoid, where A is the adjacency matrix of an irreducible finite directed graph (V, E) (see Example 2.5). There exists a topologically mixing one-sided SFT (X_B, σ_B) such that $\mathscr{G}_A \cong \mathscr{G}_B$ [29, Lemma 5.6]. Define $c : \mathscr{G}_B \to \mathbb{Z}$ by $c(x, n, y) = n$. Then c is a continuous surjective homomorphism and $\mathscr{H} = \operatorname{Ker} c$ is an AF groupoid. Since (X_B, σ_B) is topologically mixing (or equivalently the matrix B is primitive), $\mathscr{H}$ is minimal and $M(\mathscr{H})$ is a singleton. One can check that condition (1) of the proposition above is satisfied (the real number λ is the inverse of the Perron eigenvalue of B). Consequently, $\mathscr{G}_B$ satisfies the AH conjecture, and so does $\mathscr{G}_A$. (Indeed, we know that SFT groupoids have the strong AH property, see Example 4.8 (3).)

By the same technique as the example above, we can prove the following.

Theorem 7.6 ([29, Theorem 5.8]) *Let $\mathscr{G} = \mathscr{G}_{A_1} \times \mathscr{G}_{A_2} \times \cdots \times \mathscr{G}_{A_n}$ be a product groupoid of finitely many SFT groupoids. Then, $\mathscr{G}$ satisfies the AH conjecture (Conjecture 4.7).*

8 Various Examples

8.1 Minimal $\mathbb{Z}$-Actions

In this subsection, we would like to review the results about étale groupoids of minimal $\mathbb{Z}$-actions on Cantor sets. Recall that these groupoids are almost finite (Proposition 6.2). We identify a $\mathbb{Z}$-action $\mathbb{Z} \curvearrowright X$ with a homeomorphism on X.

Theorem 8.1 ([14, Theorem 2.4]) *For $i = 1, 2$, let φ_i be a minimal homeomorphism on a Cantor set X_i. The following are equivalent.*

1. *The étale groupoids $\mathscr{G}_{\varphi_1}$ and $\mathscr{G}_{\varphi_2}$ are isomorphic to each other.*
2. *φ_1 is flip conjugate to φ_2.*

Proof We present a sketchy proof. (2)⇒(1) is obvious. Suppose that $\pi : \mathscr{G}_{\varphi_1} \to \mathscr{G}_{\varphi_2}$ is an isomorphism. We may assume $X = X_1 = X_2$ and $\pi|X$ is the identity. Hence there exists a continuous map $c : \mathbb{Z} \times X \to \mathbb{Z}$ such that $\pi(n, x) = (c(n, x), x)$. One has $\varphi_1(x) = \varphi_2^{c(1,x)}(x)$ for all $x \in X$. Let $C = \max\{|c(1, x)| \mid x \in X\} < \infty$.

It is easy to check that $c(n+m,x) = c(n,\varphi_1^m(x)) + c(m,x)$ holds for all $m,n \in \mathbb{Z}$ and $x \in X$. For each $x \in X$, the map $m \mapsto c(m,x)$ is a bijection on $\mathbb{Z}$, and $|c(m+1,x) - c(m,x)| \le C$. Therefore, we have two possibilities: $c(m,x) \to +\infty$ as $m \to +\infty$ or $c(m,x) \to -\infty$ as $m \to +\infty$. Without loss of generality, we may assume that the former holds. Define $b : X \to \mathbb{Z}$ by

$$b(x) = \lim_{N\to\infty} \#\{n \in \mathbb{N} \mid c(n,x) \le N\} - N.$$

Indeed, there exists $M \in \mathbb{N}$ such that if $c(n,x) > M$ then $n > 0$. So, $b(x) = \#\{n \in \mathbb{N} \mid c(n,x) \le N\} - N$ if $N \ge M$. Especially, b is continuous. Define a continuous map $\gamma : X \to X$ by $\gamma(x) = \varphi_2^{-b(x)}(x)$. For sufficiently large N, we can verify

$$\begin{aligned} b(\varphi_1(x)) &= \#\{n \in \mathbb{N} \mid c(n,\varphi_1(x)) \le N\} - N \\ &= \#\{n \in \mathbb{N} \mid c(n+1,x) \le N + c(1,x)\} - N \\ &= \#\{n \in \mathbb{N} \mid c(n,x) \le N + c(1,x)\} - 1 - N \\ &= \#\{n \in \mathbb{N} \mid c(n,x) \le N + c(1,x)\} - (N + c(1,x)) + c(1,x) - 1 \\ &= b(x) + c(1,x) - 1, \end{aligned}$$

which implies $\varphi_2 \circ \gamma = \gamma \circ \varphi_1$. Since every orbit of φ_1 (or φ_2) is infinite, one can show that γ is a homeomorphism. It follows that φ_1 is conjugate to φ_2. □

Theorem 8.2 *Let φ be a minimal homeomorphism on a Cantor set X.*

1. *$H_0(\mathcal{G}_\varphi)$ is isomorphic to $C(X,\mathbb{Z})/\{f - f \circ \varphi \mid f \in C(X,\mathbb{Z})\}$, $H_1(\mathcal{G}_\varphi) \cong \mathbb{Z}$ and $H_n(\mathcal{G}_\varphi) = 0$ for $n \ge 2$.*
2. *$D([[\mathcal{G}_\varphi]])$ is simple.*
3. *The index map $I : [[\mathcal{G}_\varphi]] \to H_1(\mathcal{G}_\varphi)$ is surjective.*
4. *$[[\mathcal{G}_\varphi]]_0/D([[\mathcal{G}_\varphi]])$ is isomorphic to $H_0(\mathcal{G}_\varphi) \otimes \mathbb{Z}_2$.*

Proof

(1) is obvious because $H_n(\mathcal{G}_\varphi)$ is isomorphic to the group homology

$$H_n(\mathbb{Z}, C(X,\mathbb{Z}))$$

(see Example 3.3 (2)).

(2) is a special case of Theorem 6.5 (1).

(3) is clear because of $I(\varphi) = 1 \in \mathbb{Z} = H_1(\mathcal{G}_\varphi)$. (It can be also viewed as a special case of Theorem 6.5 (2)).

We give a brief explanation of (4). See [26] for a detailed proof. Fix a point $y \in X$. Define a subgroupoid $\mathcal{H} \subset \mathcal{G}_\varphi = \mathbb{Z} \times X$ by

$$\mathcal{H} = \mathcal{G}_\varphi \setminus \{(n,\varphi^m(y)) \mid m \le 0 < n+m \;\text{ or }\; n+m \le 0 < m\}.$$

Then $\mathscr{H}$ is open and becomes a minimal AF subgroupoid with the relative topology [16, Theorem 4.3]. Evidently $[[\mathscr{H}]]$ is a subgroup of $[[\mathscr{G}_\varphi]]$. As observed in Example 4.2, the topological full group $[[\mathscr{H}]]$ of the AF groupoid $\mathscr{H}$ is an increasing union of subgroups isomorphic to finite direct sums of symmetric groups, and the inclusion map of each step is given by the edge set E_n. It follows that $D([[\mathscr{H}]])$ is an increasing union of subgroups isomorphic to finite direct sums of alternating groups. Since $\mathscr{H}$ is minimal, we can easily deduce that $D([[\mathscr{H}]])$ is simple. From this, with some extra work, we get the simplicity of $D([[\mathscr{G}_\varphi]])$. On the other hand, $[[\mathscr{H}]]/D([[\mathscr{H}]])$ is isomorphic to an inductive limit of finite direct sums of $\mathbb{Z}_2$, and the connecting map of each step is given by the edge set E_n. Hence one has $[[\mathscr{H}]]/D([[\mathscr{H}]]) \cong H_0(\mathscr{H}) \otimes \mathbb{Z}_2$. The inclusion map $\mathscr{H} \to \mathscr{G}$ induces $H_0(\mathscr{H}) \cong H_0(\mathscr{G}_\varphi)$, which implies $[[\mathscr{H}]]/D([[\mathscr{H}]]) \cong H_0(\mathscr{G}_\varphi) \otimes \mathbb{Z}_2$. The index map I kills all the elements of finite order, and so $[[\mathscr{H}]]$ is contained in $[[\mathscr{G}_\varphi]]_0$. Then, with some extra work, we can show $[[\mathscr{G}_\varphi]]_0/D([[\mathscr{G}_\varphi]]) \cong H_0(\mathscr{G}_\varphi) \otimes \mathbb{Z}_2$. □

For $\alpha \in [[\mathscr{G}_\varphi]]$, $I(\alpha) \in \mathbb{Z}$ is computed as follows (see [15, Section 5] for the detailed argument). Fix a φ-invariant probability measure $\mu \in M(\mathscr{G}_\varphi)$. By Example 4.3, there exists a continuous map $n : X \to \mathbb{Z}$ such that $\alpha(x) = \varphi^{n(x)}(x)$ for all $x \in X$. Define $I' : [[\mathscr{G}_\varphi]] \to \mathbb{R}$ by

$$I'(\alpha) = \int_X n(x)\, d\mu(x).$$

It is easy to see that I' is a homomorphism and $I'(\varphi) = 1$. By [15, Lemma 5.3] (or [26, Lemma 4.1]), the group $[[\mathscr{G}_\varphi]]_0$ is generated by elements of finite order. Therefore the kernel of I' contains $[[\mathscr{G}_\varphi]]_0$. So, we can conclude $I = I'$.

We say that a homeomorphism φ on a Cantor set is expansive if there exists a continuous map f from X to a finite set A such that $f(\varphi^n(x)) = f(\varphi^n(y))$ for all $n \in \mathbb{Z}$ implies $x = y$.

Theorem 8.3 ([26, Theorem 5.4, Theorem 5.7]) *Let φ be a minimal homeomorphism on a Cantor set X.*

1. *$D([[\mathscr{G}_\varphi]])$ is finitely generated if and only if φ is expansive.*
2. *$D([[\mathscr{G}_\varphi]])$ never be finitely presented.*

Notice that (1) is a special case of [33, Theorem 5.6]. In particular, the same statement holds for any free minimal action $\varphi : \mathbb{Z}^N \curvearrowright X$. (2) was shown in [26] by expressing (X, φ) as a decreasing 'intersection' of two-sided shifts of finite type. Later, Grigorchuk and Medynets [18] proved that $[[\mathscr{G}_\varphi]]$ is locally embeddable into finite groups, which implies that $D([[\mathscr{G}_\varphi]])$ never be finitely presented. We do not know if $D([[\mathscr{G}_\varphi]])$ can be finitely presented when φ is a free minimal action of $\mathbb{Z}^N$.

Juschenko and Monod obtained the following remarkable result.

Theorem 8.4 ([20, Theorem A]) *Let φ be a minimal homeomorphism on a Cantor set X. Then $[[\mathscr{G}_\varphi]]$ is amenable.*

In the proof of this theorem, the notion of extensive amenability plays the central role. This property was first introduced (without a name) in [20], and studied further in [19, 21].

We recall the definition of extensive amenability from [19, Definition 1.1]. Let $\alpha : G \curvearrowright Z$ be an action of a discrete group G on a set Z. We denote by $\mathscr{P}_f(Z)$ the set of all finite subsets of Z. The collection $\mathscr{P}_f(Z)$ is an abelian group for the operation Δ of symmetric difference. The action α naturally extends to $\alpha : G \curvearrowright \mathscr{P}_f(Z)$. We say that $\alpha : G \curvearrowright Z$ is extensively amenable if there exists a G-invariant mean (i.e. finitely additive probability measure) m on $\mathscr{P}_f(Z)$ such that $m(\{F \in \mathscr{P}_f(Z) \mid E \subset F\}) = 1$ for any $E \in \mathscr{P}_f(Z)$. In [20, Lemma 3.1], it was shown that $\alpha : G \curvearrowright Z$ is extensively amenable if and only if the action of $\mathscr{P}_f(Z) \rtimes G$ on $\mathscr{P}_f(Z)$ admits an invariant mean.

We denote by $W(\mathbb{Z})$ the group of all permutations g of $\mathbb{Z}$ for which the quantity $\sup\{|g(j) - j| \mid j \in \mathbb{Z}\}$ is finite. In [20, Theorem C], it was shown that the natural action $W(\mathbb{Z}) \curvearrowright \mathbb{Z}$ is extensively amenable. (This part is technically quite hard.) It follows that the action of $\mathscr{P}_f(\mathbb{Z}) \rtimes W(\mathbb{Z})$ on $\mathscr{P}_f(\mathbb{Z})$ admits an invariant mean.

Let φ be a minimal homeomorphism on a Cantor set X. We would like to show that $[[\mathscr{G}_\varphi]]$ is amenable. Fix a point $x \in X$. We can define a map $\pi : [[\mathscr{G}_\varphi]] \to W(\mathbb{Z})$ so that $\gamma(\varphi^j(x)) = \varphi^{\pi(\gamma)(j)}(x)$ for every $\gamma \in [[\mathscr{G}_\varphi]]$ and $j \in \mathbb{Z}$. The map π is an injective homomorphism. Define a map $\tilde{\pi} : [[\mathscr{G}_\varphi]] \to \mathscr{P}_f(\mathbb{Z}) \rtimes W(\mathbb{Z})$ by $\tilde{\pi}(\gamma) = (\mathbb{N}\Delta\pi(\gamma)(\mathbb{N}), \pi(\gamma))$ for $\gamma \in [[\mathscr{G}_\varphi]]$. It is routine to check that $\tilde{\pi}$ is an injective homomorphism. Since the action of $\mathscr{P}_f(\mathbb{Z}) \rtimes W(\mathbb{Z})$ on $\mathscr{P}_f(\mathbb{Z})$ admits an invariant mean, in order to show the amenability of $[[\mathscr{G}_\varphi]]$, it suffices to prove that the stabiliser in $\tilde{\pi}([[\mathscr{G}_\varphi]])$ of E is amenable for any $E \in \mathscr{P}_f(\mathbb{Z})$. By [20, Lemma 4.1], the stabiliser is locally finite, and hence amenable. This completes the proof.

In [19, 21], the notion of extensive amenability is used to prove amenability of various kinds of groups. Among others, it was shown that all subgroups of the group of interval exchange transformations that have angular components of rational rank ≤ 2 are amenable [19, Theorem 5.1]. In particular, when $\varphi : \mathbb{Z}^2 \curvearrowright X$ is a free minimal action arising from two irrational rotations on the circle (see [17, Example 30]), the topological full group $[[\mathscr{G}_\varphi]]$ is amenable. On the other hand, it is known that there exists a free minimal action $\varphi : \mathbb{Z}^2 \curvearrowright X$ on a Cantor set such that $[[\mathscr{G}_\varphi]]$ contains the non-abelian free group [11]. It may be a rather complicated problem to determine when $[[\mathscr{G}_\varphi]]$ is amenable for $\varphi : \mathbb{Z}^2 \curvearrowright X$.

8.2 Shifts of Finite Type

In this subsection, we would like to review the results about étale groupoids of one-sided shifts of finite type (see Definition 2.5). Note that these groupoids are purely infinite and minimal [28, Lemma 6.1].

Theorem 8.5 ([25, Theorem 3.6]) *Let (X_A, σ_A) and (X_B, σ_B) be two irreducible one-sided shifts of finite type. The following conditions are equivalent.*

1. *(X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent.*
2. *The étale groupoids $\mathcal{G}_A$ and $\mathcal{G}_B$ are isomorphic.*
3. *There exists an isomorphism $\varphi : C_r^*(\mathcal{G}_A) \to C_r^*(\mathcal{G}_B)$ such that $\varphi(C(X_A)) = C(X_B)$.*
4. *There exists an isomorphism $\pi : H_0(\mathcal{G}_A) \to H_0(\mathcal{G}_B)$ such that $\pi([1_{X_A}]) = [1_{X_B}]$ and* $\mathrm{sgn}(\det(\mathrm{id} - A)) = \mathrm{sgn}(\det(\mathrm{id} - B))$.

For the definition of continuous orbit equivalence, see [25, Section 2.1]. As mentioned in Example 3.3 (3), we have

$$H_n(\mathcal{G}_A) \cong \begin{cases} \mathrm{Coker}(\mathrm{id} - A^t) & n = 0 \\ \mathrm{Ker}(\mathrm{id} - A^t) & n = 1 \\ 0 & n \geq 2. \end{cases}$$

The element $[1_{X_A}] \in H_0(\mathcal{G}_A)$ corresponds to the equivalence class of $(1, 1, \dots, 1)$ in $\mathrm{Coker}(\mathrm{id} - A^t)$.

Proof We present a sketchy proof of the theorem above. The equivalence (1) $\iff$ (2) directly follows from the definition. The equivalence (2) $\iff$ (3) is a special case of Theorem 2.1.

(4)$\Rightarrow$(3). Let $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ denote the two-sided shifts of finite type. By the result of Franks [13], $H_0(\mathcal{G}_A) \cong H_0(\mathcal{G}_B)$ and

$$\mathrm{sgn}(\det(\mathrm{id} - A)) = \mathrm{sgn}(\det(\mathrm{id} - B))$$

imply that $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent. It follows from [8, Theorem 4.1] that there exists an isomorphism $\varphi : C_r^*(\mathcal{G}_A) \otimes \mathcal{K} \to C_r^*(\mathcal{G}_B) \otimes \mathcal{K}$ such that $\varphi(C(X_A) \otimes \mathcal{C}) = C(X_B) \otimes \mathcal{C}$, where $\mathcal{C} \cong c_0(\mathbb{Z})$ is the maximal abelian subalgebra of $\mathcal{K}$ consisting of diagonal operators. Since the isomorphism $\pi : H_0(\mathcal{G}_A) \to H_0(\mathcal{G}_B)$ carries $[1_{X_A}]$ to $[1_{X_B}]$, a suitable modification of φ yields the desired isomorphism.

(2)$\Rightarrow$(4). Clearly, $\mathcal{G}_A \cong \mathcal{G}_B$ implies $(H_0(\mathcal{G}_A), [1_{X_A}]) \cong (H_0(\mathcal{G}_B), [1_{X_B}])$. The ordered cohomology group of $(\bar{X}_A, \bar{\sigma}_A)$, introduced by Boyle and Handelman [2], is the abelian group $\bar{H}^A = C(\bar{X}_A, \mathbb{Z}) / \{\xi - \xi \circ \bar{\sigma}_A \mid \xi \in C(\bar{X}_A, \mathbb{Z})\}$ with the positive cone $\bar{H}^A_+ = \{[\xi] \in \bar{H}^A \mid \xi \geq 0\}$. We can prove that $\mathcal{G}_A \cong \mathcal{G}_B$ implies $(\bar{H}^A, \bar{H}^A_+) \cong (\bar{H}^B, \bar{H}^B_+)$ (see [25, Theorem 3.5]). Then, by [2, Theorem 1.12], $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent. As a result, we obtain $\mathrm{sgn}(\det(\mathrm{id} - A)) = \mathrm{sgn}(\det(\mathrm{id} - B))$. $\square$

Theorem 8.6 *Let $\mathscr{G}_A$ be an SFT groupoid.*

1. $D([[\mathscr{G}_A]])$ *is simple.*
2. *The index map* $I : [[\mathscr{G}_A]] \to H_1(\mathscr{G}_A)$ *is surjective.*
3. $[[\mathscr{G}_A]]_0/D([[\mathscr{G}_A]])$ *is isomorphic to* $H_0(\mathscr{G}_A) \otimes \mathbb{Z}_2$.

Proof (1) and (2) immediately follow from Theorem 7.3 (1) and (2).

(3). By Example 7.5,

$$H_0(\mathscr{G}_A) \otimes \mathbb{Z}_2 \xrightarrow{\;j\;} [[\mathscr{G}_A]]_{\mathrm{ab}} \longrightarrow H_1(\mathscr{G}_A) \longrightarrow 0$$

is exact. It suffices to show that j is injective and has a left inverse. This was shown in [28, Section 6.6], by using a finite presentation of $[[\mathscr{G}_A]]$. Here, we would like to describe another approach.

As mentioned in Example 4.4, when A is a 1×1 matrix $[n]$, $[[\mathscr{G}_{[n]}]]$ is the Higman-Thompson group $V_{n,1}$. It is well-known that the abelianization of $V_{n,1}$ is trivial if n is even, and is $\mathbb{Z}_2$ if n is odd. Suppose that an SFT groupoid $\mathscr{G}_A$ is given. Let $\varphi : H_0(\mathscr{G}_A) \to \mathbb{Z}_2 \cong H_0(\mathscr{G}_{[3]})$ be a homomorphism. Choose a nonempty clopen set $Y \subset X_{[3]}$ so that $[1_Y] = \varphi([1_{X_A}])$. Set $\mathscr{H} = \mathscr{G}_{[3]}|Y$. We have $[[\mathscr{H}]]_{\mathrm{ab}} \cong \mathbb{Z}_2$. By Theorem 8.7, we can find an embedding $\pi : C_r^*(\mathscr{G}_A) \to C_r^*(\mathscr{H})$ which induces

$$\begin{array}{ccccccccc}
1 & \longrightarrow & U(C(\mathscr{G}_A^{(0)})) & \longrightarrow & N(C(X_A), C_r^*(\mathscr{G}_A)) & \longrightarrow & [[\mathscr{G}_A]] & \longrightarrow & 1 \\
 & & {\scriptstyle\pi}\downarrow & & {\scriptstyle\pi}\downarrow & & \downarrow & & \\
1 & \longrightarrow & U(C(Y)) & \longrightarrow & N(C(Y), C_r^*(\mathscr{H})) & \longrightarrow & [[\mathscr{H}]] & \longrightarrow & 1,
\end{array}$$

and $\varphi([1_P]) = [\pi(1_P)]$ for any clopen set $P \subset X_A$. Let us denote the embedding $[[\mathscr{G}_A]] \to [[\mathscr{H}]]$ (and also the induced homomorphism $[[\mathscr{G}_A]]_{\mathrm{ab}} \to [[\mathscr{H}]]_{\mathrm{ab}}$) by π_*. Let U be a compact open $\mathscr{G}_A$-set satisfying $r(U) \cap s(U) = \emptyset$. Define $\tau_U \in [[\mathscr{G}_A]]$ by

$$\tau_U(x) = \begin{cases} \theta(U)(x) & x \in s(U) \\ \theta(U^{-1})(x) & x \in r(U) \\ x & \text{otherwise.} \end{cases}$$

Then $j([1_{s(U)}] \otimes \bar{1})$ equals the equivalence class of τ_U in $[[\mathscr{G}_A]]_{\mathrm{ab}}$. It is easy to see that the equivalence class of $\pi_*(\tau_U)$ equals $j(\varphi([1_{s(U)}]) \otimes \bar{1})$. Thus, $\pi_*(j([1_{s(U)}] \otimes \bar{1})) = j(\varphi([1_{s(U)}]) \otimes \bar{1})$. Hence $\pi_* \circ j = j \circ (\varphi \otimes \mathrm{id})$. Since the homomorphism $\varphi : H_0(\mathscr{G}_A) \to \mathbb{Z}_2 \cong H_0(\mathscr{G}_{[3]})$ was arbitrary, we obtain the desired conclusion. □

In the proof above, we use the following embedding theorem.

Theorem 8.7 ([29, Proposition 5.14]) *Let $\mathscr{G}_A$ be an SFT groupoid and let $\mathscr{H}$ be a minimal, purely infinite étale groupoid. Suppose that $\varphi : H_0(\mathscr{G}_A) \to H_0(\mathscr{H})$ is a homomorphism satisfying $\varphi([1_{X_A}]) = [1_{\mathscr{H}^{(0)}}]$. Then there exists a unital homomorphism $\pi : C^*_r(\mathscr{G}_A) \to C^*_r(\mathscr{H})$ such that the following hold.*

1. $\pi(C(X_A)) \subset C(\mathscr{H}^{(0)})$.
2. *For any compact open $\mathscr{G}_A$-set U, there exists a compact open $\mathscr{H}$-set V such that $\pi(1_U) = 1_V$.*
3. *For any clopen set $P \subset X_A$, $[\pi(1_P)] = \varphi([1_P])$ in $H_0(\mathscr{H})$.*

In particular, π induces an embedding of $[[\mathscr{G}_A]]$ into $[[\mathscr{H}]]$.

The proof of this theorem uses the fact that the Cuntz-Krieger algebra $C^*_r(\mathscr{G}_A)$ is characterized as the universal C^*-algebra generated by partial isometries subject to the Cuntz-Krieger relations [8].

As for finiteness condition, the following is known.

Theorem 8.8 ([28, Section 6]) *Let $\mathscr{G}_A$ be an SFT groupoid.*

1. $[[\mathscr{G}_A]]$ *is of type F_∞. (In particular, it is finitely presented.)*
2. $[[\mathscr{G}_A]]_0$ *and* $D([[\mathscr{G}_A]])$ *are finitely generated.*

For the Higman-Thompson groups $V_{n,r}$ (and also $F_{n,r}$, $T_{n,r}$), Brown [6] proved that they are of type F_∞. The theorem above is a generalization of this result, and its proof uses Brown's criterion [6, Corollary 3.3].

As mentioned in Sect. 7, $[[\mathscr{G}_A]]$ is not amenable, because $\mathscr{G}_A$ is purely infinite. But, $[[\mathscr{G}_A]]$ has a weaker version of the amenability.

Theorem 8.9 ([28, Theorem 6.7]) *Let $\mathscr{G}_A$ be an SFT groupoid. The topological full group $[[\mathscr{G}_A]]$ has the Haagerup property.*

Farley [12] proved that the Higman-Thompson group $V_{2,1}$ has the Haagerup property. In [28], the Haagerup property of $[[\mathscr{G}_A]]$ was shown by modifying the argument of [12]. But, making use of Theorem 8.7, we can embed $[[\mathscr{G}_A]]$ to $[[\mathscr{G}_{[2]}]] \cong V_{2,1}$, and so the Haagerup property of $[[\mathscr{G}_A]]$ follows immediately.

Soficity and exactness of the Higman-Thompson groups (or $[[\mathscr{G}_A]]$) is an important open problem.

8.3 *Products of Shifts of Finite Type*

In this subsection, we would like to review the results about product groupoids $\mathscr{G} = \mathscr{G}_{A_1} \times \mathscr{G}_{A_2} \times \cdots \times \mathscr{G}_{A_n}$ of SFT groupoids. Evidently, these groupoids are purely infinite and minimal. (In general, if $\mathscr{G}$ is purely infinite, then $\mathscr{G} \times \mathscr{H}$ is purely infinite, too.) The groupoid C^*-algebra $C^*_r(\mathscr{G})$ is isomorphic to the tensor product $C^*_r(\mathscr{G}_{A_1}) \otimes C^*_r(\mathscr{G}_{A_2}) \otimes \cdots \otimes C^*_r(\mathscr{G}_{A_n})$.

In what follows, for an irreducible one-sided SFT (X_A, σ_A), the equivalence class of 1_{X_A} in $H_0(\mathscr{G}_A)$ is denoted by u_A.

Theorem 8.10 ([29, Theorem 5.12]) *Let $\mathcal{G} = \mathcal{G}_{A_1} \times \mathcal{G}_{A_2} \times \cdots \times \mathcal{G}_{A_m}$ and $\mathcal{H} = \mathcal{G}_{B_1} \times \mathcal{G}_{B_2} \times \cdots \times \mathcal{G}_{B_n}$ be product groupoids of SFT groupoids. Then $\mathcal{G} \cong \mathcal{H}$ if and only if the following are satisfied.*

1. *$m = n$.*
2. *There exist a permutation σ of $\{1, 2, \ldots, n\}$ and isomorphisms $\varphi_i : H_0(\mathcal{G}_{A_i}) \to H_0(\mathcal{G}_{B_{\sigma(i)}})$ such that $\det(\mathrm{id} - A_i) = \det(\mathrm{id} - B_{\sigma(i)})$ and*

$$(\varphi_1 \otimes \varphi_2 \otimes \cdots \otimes \varphi_n)(u_{A_1} \otimes u_{A_2} \otimes \cdots \otimes u_{A_n}) = u_{B_{\sigma(1)}} \otimes u_{B_{\sigma(2)}} \otimes \cdots \otimes u_{B_{\sigma(n)}}.$$

In particular, $\mathcal{G}_{A_i}$ and $\mathcal{G}_{B_{\sigma(i)}}$ are Morita equivalent.

Theorem 8.11 ([29]) *Let $\mathcal{G} = \mathcal{G}_{A_1} \times \mathcal{G}_{A_2} \times \cdots \times \mathcal{G}_{A_n}$ be a product groupoid of SFT groupoids.*

1. *$H_k(\mathcal{G})$ is isomorphic to*

$$\left(\mathbb{Z}^{\binom{n-1}{k}} \otimes H_0(\mathcal{G}_{A_1}) \otimes H_0(\mathcal{G}_{A_2}) \otimes \cdots \otimes H_0(\mathcal{G}_{A_n})\right)$$
$$\oplus \left(\mathbb{Z}^{\binom{n-1}{k-1}} \otimes H_1(\mathcal{G}_{A_1}) \otimes H_1(\mathcal{G}_{A_2}) \otimes \cdots \otimes H_1(\mathcal{G}_{A_n})\right),$$

where $\binom{n}{k}$ denote the binomial coefficients and they are understood as zero unless $0 \leq k \leq n$. The equivalence class of the constant function $1_{\mathcal{G}^{(0)}}$ in $H_0(\mathcal{G}) = H_0(\mathcal{G}_{A_1}) \otimes \cdots \otimes H_0(\mathcal{G}_{A_n})$ is $u_{A_1} \otimes \cdots \otimes u_{A_n}$.
2. *$\mathcal{G}$ satisfies the HK conjecture.*
3. *$D([[\mathcal{G}]])$ is simple.*
4. *The index map $I : [[\mathcal{G}]] \to H_1(\mathcal{G})$ is surjective.*
5. *$\mathcal{G}$ satisfies the AH conjecture.*

Proof (1) is obtained from the Künneth theorem (Theorem 3.4).

(2) is an immediate consequence of Theorem 3.4 and the Künneth theorem for K-groups of C^*-algebras.

(3) and (4) readily follow from Theorem 7.3 (1) and (2).

(5) is shown by an inductive application of Proposition 7.4. See also Example 7.5 and Theorem 7.6. □

The topological full group of $\mathcal{G} = \mathcal{G}_{A_1} \times \mathcal{G}_{A_2} \times \cdots \times \mathcal{G}_{A_n}$ is a generalization of the higher dimensional Thompson groups $nV_{k,r}$. Brin introduced the notion of higher dimensional Thompson groups $nV_{k,r}$ in [3, Section 4.2]. These groups can be considered as an n-dimensional analogue of the Higman-Thompson group $V_{k,r} = 1V_{k,r}$. Brin proved that $V_{k,r}$ and $2V_{2,1}$ are not isomorphic, and that $2V_{2,1}$ is finitely presented. He also proved that $nV_{2,1}$ is simple for all $n \in \mathbb{N}$ in [4]. Dicks and Martínez-Pérez [9] proved that $nV_{k,r} \cong n'V_{k',r'}$ if and only if $n = n'$, $k = k'$ and $\gcd(k-1, r) = \gcd(k'-1, r')$.

Define an $r \times r$ matrix $A_{k,r}$ by

$$A_{k,r} = \begin{bmatrix} 0 & 0 & \dots & 0 & k \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}.$$

The topological full group $[[\mathscr{G}_{A_{k,r}}]]$ of the SFT groupoid $A_{k,r}$ is naturally isomorphic to the Higman-Thompson group $V_{k,r}$ (see [28, Section 6.7.1]). By Example 3.3 (3), $H_0(\mathscr{G}_{A_{k,r}}) \cong \mathbb{Z}_{k-1}$, $H_n(\mathscr{G}_{A_{k,r}}) = 0$ for $n \geq 1$ and $u_{A_{k,r}}$ corresponds to $\bar{r} \in \mathbb{Z}_{k-1}$. It is not so hard to see that the higher dimensional Thompson group $nV_{k,r}$ is isomorphic to the topological full group $[[\mathscr{G}]]$ of the product groupoid

$$\mathscr{G} = \mathscr{G}_{A_{k,r}} \times \overbrace{\mathscr{G}_{A_{k,1}} \times \dots \times \mathscr{G}_{A_{k,1}}}^{n-1}.$$

It follows from Theorem 8.11 (3) that the commutator subgroup $D([[\mathscr{G}]])$ is simple. By Theorem 8.11 (1), we get $H_l(\mathscr{G}) \cong (\mathbb{Z}_{k-1})^{\binom{n-1}{l}}$. Therefore, Theorem 8.11 (5) tells us that $[[\mathscr{G}]]$ is simple if and only if $k = 2$. This reproves the result of Brin [4]. Also, by applying Theorem 8.10, we get a new proof of the classification theorem by Dicks and Martínez-Pérez [9].

As for the cohomological finiteness condition, Martínez-Pérez, Matucci and Nucinkis [24] proved that $nV_{k,1}$ (and many other relatives) are of type F_∞. We do not know if the same holds for topological full groups of product groupoids of SFT groupoids $\mathscr{G} = \mathscr{G}_{A_1} \times \mathscr{G}_{A_2} \times \dots \times \mathscr{G}_{A_n}$.

In [29, Section 5.5], we completely determined the abelianization of the topological full group of $\mathscr{G} = \mathscr{G}_{A_1} \times \mathscr{G}_{A_2} \times \dots \times \mathscr{G}_{A_n}$, but here we do not state the result precisely because it would be quite complicated. Instead, let us consider a special case, namely products of one-sided full shifts. Let $k : \{1, 2, \dots, n\} \to \mathbb{N} \setminus \{1\}$ be a map. Set

$$\mathscr{G} = \mathscr{G}_{[k(1)]} \times \mathscr{G}_{[k(2)]} \times \dots \times \dots \mathscr{G}_{[k(n)]}.$$

Let $g = \gcd\{k(i) - 1 \mid i = 1, 2, \dots, n\}$. Then $H_0(\mathscr{G}) \cong \mathbb{Z}_g$ and $H_1(\mathscr{G}) \cong (\mathbb{Z}_g)^{n-1}$. By Theorem 8.11 (5),

$$\mathbb{Z}_g \otimes \mathbb{Z}_2 \xrightarrow{\ j\ } [[\mathscr{G}]]_{\mathrm{ab}} \xrightarrow{\ I\ } (\mathbb{Z}_g)^{n-1} \longrightarrow 0$$

is exact.

Theorem 8.12 ([29, Theorem 5.23]) *Let $\mathscr{G}$ be as above.*

1. *If $k(i)$ is even for some i, then $[[\mathscr{G}]]_{\mathrm{ab}} \cong (\mathbb{Z}_g)^{n-1}$.*
2. *If $k(i)$ is odd for all i and $\#\{i \mid k(i) \in 4\mathbb{Z}+3\} \leq 1$, then $[[\mathscr{G}]]_{\mathrm{ab}} \cong \mathbb{Z}_2 \oplus (\mathbb{Z}_g)^{n-1}$.*
3. *If $k(i)$ is odd for all i and $\#\{i \mid k(i) \in 4\mathbb{Z}+3\} = 2$, then $[[\mathscr{G}]]_{\mathrm{ab}} \cong \mathbb{Z}_{2g} \oplus (\mathbb{Z}_g)^{n-2}$.*
4. *If $k(i)$ is odd for all i and $\#\{i \mid k(i) \in 4\mathbb{Z}+3\} \geq 3$, then $[[\mathscr{G}]]_{\mathrm{ab}} \cong (\mathbb{Z}_g)^{n-1}$. In particular, $\mathscr{G}$ does not have the strong AH property.*

Proof

(1). Since g is odd, $\mathbb{Z}_g \otimes \mathbb{Z}_2 = 0$. So $[[\mathscr{G}]]_{\mathrm{ab}} \cong (\mathbb{Z}_g)^{n-1}$.

(2). Let us consider the case that $k(i) \in 4\mathbb{Z}+1$ for all $i = 1, 2, \dots, n$. Let $\mathscr{H}$ be the direct product of n copies of $\mathscr{G}_{[5]}$. By [29, Lemma 5.19 (1)], there exists a homomorphism $\rho : [[\mathscr{H}]]_{\mathrm{ab}} \to \mathbb{Z}_2$ such that $\rho \circ j$ is nonzero. (In [29, Lemma 5.19 (1)], $[[\mathscr{H}]]$ is embedded into a group (named $W_{n,k}$), and its generators and relations are explicitly written down. Using them, one can obtain the homomorphism ρ to $\mathbb{Z}_2$.) For each $i = 1, 2, \dots, n$, we define a homomorphism $\varphi_i : H_0(\mathscr{G}_{[k(i)]}) \to H_0(\mathscr{G}_{[5]})$ by $\varphi_i(\bar{1}) = \bar{1}$. Applying Theorem 8.7 to each φ_i, we get an embedding $\pi : [[\mathscr{G}]] \to [[\mathscr{H}]]$. In the same way as the proof of Theorem 8.6 (3), one can conclude that $\rho \circ \pi \circ j$ is nonzero. Thus, $[[\mathscr{G}]]_{\mathrm{ab}} \cong \mathbb{Z}_2 \oplus (\mathbb{Z}_g)^{n-1}$.

When $k(i)$ is odd for all i and $\#\{i \mid k(i) \in 4\mathbb{Z}+3\} = 1$, the same argument works by using [29, Lemma 5.19 (2)].

(3). Almost the same argument as above works, by using [29, Lemma 5.19 (3)]. But, the range of the homomorphism ρ becomes $\mathbb{Z}_4$. Thus, the map $j : H_0(\mathscr{G}) \otimes \mathbb{Z}_2 \to [[\mathscr{G}]]_{\mathrm{ab}}$ is injective but does not have a right inverse. Hence, we can conclude $[[\mathscr{G}]]_{\mathrm{ab}} \cong \mathbb{Z}_{2g} \oplus (\mathbb{Z}_g)^{n-2}$.

(4). For simplicity, we assume $n = 3$ and $k(1) = k(2) = k(3) = 3$. It suffices to show that $j : H_0(\mathscr{G}) \otimes \mathbb{Z}_2 \to [[\mathscr{G}]]_{\mathrm{ab}}$ is zero.

Let $X_{[3]} = \{0, 1, 2\}^{\mathbb{N}}$ and let $(X_{[3]}, \sigma_{[3]})$ be the full shift over three symbols. We define the clopen set $C_i \subset X_{[3]}$ by

$$C_i = \{(x_n)_{n\in\mathbb{N}} \in X_{[3]} \mid x_1 = i\}.$$

Define a compact open $\mathscr{G}_{[3]}$-set $U_i \subset \mathscr{G}_{[3]}$ by

$$U_i = \{(x, 1, y) \in \mathscr{G}_{[3]} \mid x \in C_i,\ \sigma_{[3]}(x) = y\}.$$

Let $t \in [[\mathscr{G}]]_{\mathrm{ab}}$ be the image of the generator of $H_0(\mathscr{G}) \otimes \mathbb{Z}_2 \cong \mathbb{Z}_2$. We would like to show $t = 0$. Define $\beta_{12} \in [[\mathscr{G}]]$ by

$$\beta_{12}(x, y, z) = \theta(U_i \times U_i^{-1} \times X_{[3]})(x, y, z) \quad \text{when } y \in C_i$$

for $(x, y, z) \in \mathscr{G}^{(0)} = X_{[3]} \times X_{[3]} \times X_{[3]}$. The homeomorphism β_{12} is the so-called baker's map acting on the first and second coordinates of $X_{[3]} \times X_{[3]} \times X_{[3]}$, and its index $I(\beta_{12})$ is nonzero in $H_1(\mathscr{G}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. By [29, Lemma 5.21 (4)], one sees

$2[\beta_{12}] = t$ in $[[\mathscr{G}]]_{\mathrm{ab}}$. We can define $\beta_{23} \in [[\mathscr{G}]]$ in the same way by

$$\beta_{23}(x, y, z) = \theta(X_{[3]} \times U_i \times U_i^{-1})(x, y, z) \quad \text{when } z \in C_i.$$

Again one has $2[\beta_{23}] = t$. It is easy to see that $\beta_{12}\beta_{23}$ is equal to the baker's map acting on the first and third coordinates of $X_{[3]} \times X_{[3]} \times X_{[3]}$. Therefore, we get $2[\beta_{12}\beta_{23}] = t$. Consequently, we obtain $2t = t$, thus $t = 0$. □

Little is known about analytic properties of $[[\mathscr{G}]]$. For example, it is natural to ask if the topological full group $[[\mathscr{G}]]$ of $\mathscr{G} = \mathscr{G}_{A_1} \times \mathscr{G}_{A_2} \times \cdots \times \mathscr{G}_{A_n}$ has the Haagerup property or not.

References

1. S. Bezuglyi and K. Medynets, *Full groups, flip conjugacy, and orbit equivalence of Cantor minimal systems*, Colloq. Math. 110 (2008), 409–429.
2. M. Boyle and D. Handelman, *Orbit equivalence, flow equivalence and ordered cohomology*, Israel J. Math. 95 (1996), 169–210.
3. M. G. Brin, *Higher dimensional Thompson groups*, Geom. Dedicata 108 (2004), 163–192. arXiv:math/0406046
4. M. G. Brin, *On the baker's map and the simplicity of the higher dimensional Thompson groups nV*, Publ. Mat. 54 (2010), 433–439. arXiv:0904.2624
5. K. S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, 87. Springer-Verlag, New York-Berlin, 1982.
6. K. S. Brown, *Finiteness properties of groups*, J. Pure Appl. Algebra 44 (1987), 45–75.
7. M. Crainic and I. Moerdijk, *A homology theory for étale groupoids*, J. Reine Angew. Math. 521 (2000), 25–46.
8. J. Cuntz and W. Krieger, *A class of C^*-algebras and topological Markov chains*, Invent. Math. 56 (1980), 251–268.
9. W. Dicks and C. Martínez-Pérez, *Isomorphisms of Brin-Higman-Thompson groups*, Israel J. Math. 199 (2014), 189–218. arXiv:1112.1606
10. G. Elek and E. Szabó, *Hyperlinearity, essentially free actions and L^2-invariants. The sofic property*, Math. Ann., 332 (2005), 421–441.
11. G. Elek and N. Monod, *On the topological full group of a minimal Cantor $\mathbb{Z}^2$-system*, Proc. Amer. Math. Soc. 141 (2013), 3549–3552.
12. D. S. Farley, *Proper isometric actions of Thompson's groups on Hilbert space*, Int. Math. Res. Not. 45 (2003), 2409–2414.
13. J. Franks, *Flow equivalence of subshifts of finite type*, Ergodic Theory Dynam. Systems 4 (1984), 53–66.
14. T. Giordano, I. F. Putnam and C. F. Skau, *Topological orbit equivalence and C^*-crossed products*, J. Reine Angew. Math. 469 (1995), 51–111.
15. T. Giordano, I. F. Putnam and C. F. Skau, *Full groups of Cantor minimal systems*, Israel J. Math. 111 (1999), 285–320.
16. T. Giordano, I. F. Putnam and C. F. Skau, *Affable equivalence relations and orbit structure of Cantor dynamical systems*, Ergodic Theory Dynam. Systems 24 (2004), 441–475.
17. T. Giordano, I. F. Putnam and C. F. Skau, *The orbit structure of Cantor minimal $\mathbb{Z}^2$systems*, Operator Algebras: The Abel Symposium 2004, 145–160, Abel Symp., 1, Springer, Berlin, 2006.

18. R. Grigorchuk and K. Medynets, *On the algebraic properties of topological full groups*, Sb. Math. 205 (2014), 843–861. arXiv:1105.0719
19. K. Juschenko, N. Matte Bon, N. Monod and M. de la Salle, *Extensive amenability and an application to interval exchanges*, preprint. arXiv:1503.04977
20. K. Juschenko and N. Monod, *Cantor systems, piecewise translations and simple amenable groups*, Ann. of Math. 178 (2013), 775–787. arXiv:1204.2132
21. K. Juschenko, V. Nekrashevych and M. de la Salle, *Extensions of amenable groups by recurrent groupoids*, preprint. arXiv:1305.2637
22. F. Latrémolière and N. Ormes, *C^*-algebraic characterization of bounded orbit injection equivalence for minimal free Cantor systems*, Rocky Mountain J. Math. 42 (2012), 157–200. arXiv:0903.1881
23. X. Li, in preparation.
24. C. Martínez-Pérez, F. Matucci and B. E. A. Nucinkis, *Cohomological finiteness conditions and centralisers in generalisations of Thompson's group V*, arXiv:1309.7858
25. K. Matsumoto and H. Matui, *Continuous orbit equivalence of topological Markov shifts and Cuntz-Krieger algebras*, Kyoto J. Math. 54 (2014), 863–877. arXiv:1307.1299
26. H. Matui, *Some remarks on topological full groups of Cantor minimal systems*, Internat. J. Math. 17 (2006), 231–251. arXiv:math/0404117
27. H. Matui, *Homology and topological full groups of étale groupoids on totally disconnected spaces*, Proc. Lond. Math. Soc. (3) 104 (2012), 27–56. arXiv:0909.1624
28. H. Matui, *Topological full groups of one-sided shifts of finite type*, J. Reine Angew. Math. 705 (2015), 35–84. arXiv:1210.5800
29. H. Matui, *Étale groupoids arising from products of shifts of finite type*, preprint. arXiv:1512.01724
30. K. Medynets, *Reconstruction of orbits of Cantor systems from full groups*, Bull. London Math. Soc. 43 (2011) 1104–1110. arXiv:1006.1145
31. K. Medynets, R. Sauer and A. Thom, *Cantor systems and quasi-isometry of groups*, preprint. arXiv:1508.07578
32. V. V. Nekrashevych, *Cuntz-Pimsner algebras of group actions*, J. Operator Theory 52 (2004), 223–249.
33. V. V. Nekrashevych, *Simple groups of dynamical origin*, preprint. arXiv:1511.08241
34. N. C. Phillips, *Crossed products of the Cantor set by free minimal actions of $\mathbb{Z}^d$*, Comm. Math. Phys. 256 (2005), 1–42.
35. J. Renault, *A Groupoid Approach to C^*-algebras*, Lecture Notes in Mathematics 793, Springer, Berlin, 1980.
36. J. Renault, *Cartan subalgebras in C^*-algebras*, Irish Math. Soc. Bull. 61 (2008), 29–63. arXiv:0803.2284
37. M. Rubin, *On the reconstruction of topological spaces from their groups of homeomorphisms*, Trans. Amer. Math. Soc. 312 (1989), 487–538.

Towards a Classification of Compact Quantum Groups of Lie Type

Sergey Neshveyev and Makoto Yamashita

Abstract This is a survey of recent results on classification of compact quantum groups of Lie type, by which we mean quantum groups with the same fusion rules and dimensions of representations as for a compact connected Lie group G. The classification is based on a categorical duality for quantum group actions recently developed by De Commer and the authors in the spirit of Woronowicz's Tannaka–Krein duality theorem. The duality establishes a correspondence between the actions of a compact quantum group H on unital C*-algebras and the module categories over its representation category $\operatorname{Rep} H$. This is further refined to a correspondence between the braided-commutative Yetter–Drinfeld H-algebras and the tensor functors from $\operatorname{Rep} H$. Combined with the more analytical theory of Poisson boundaries, this leads to a classification of dimension-preserving fiber functors on the representation category of any coamenable compact quantum group in terms of its maximal Kac quantum subgroup, which is the maximal torus for the q-deformation of G if $q \neq 1$. Together with earlier results on autoequivalences of the categories $\operatorname{Rep} G_q$, this allows us to classify up to isomorphism a large class of quantum groups of G-type for compact connected simple Lie groups G. In the case of $G = \mathrm{SU}(n)$ this class exhausts all non-Kac quantum groups.

1 Introduction

The theory of quantum groups has two origins. One is the algebraic approach motivated by the quantum inverse scattering method and initiated by the discovery of quantized universal enveloping algebras by Drinfeld [10] and Jimbo [17]. The other is the operator algebraic approach developed by Woronowicz [56], which

S. Neshveyev (✉)
Department of Mathematics, University of Oslo, P.O. Box 1053, Blindern, NO-0316 Oslo, Norway
e-mail: sergeyn@math.uio.no

M. Yamashita
Department of Mathematics, Ochanomizu University, Otsuka 2-1-1, Bunkyo, 112-8610 Tokyo, Japan
e-mail: yamashita.makoto@ocha.ac.jp

T.M. Carlsen et al. (eds.), *Operator Algebras and Applications*, Abel Symposia 12,
DOI 10.1007/978-3-319-39286-8_11

stands on the philosophy of treating noncommutative C*-algebras as functions on 'noncommutative spaces', or 'pseudospaces' in Woronowicz's terminology. In both frameworks, deformations of SU(2) and, more generally, of the compact simple Lie groups appear as the most fundamental and motivating examples.

A natural question is to classify all such deformations within a reasonable system of axioms. At the infinitesimal level, this was already settled in a series of papers by Drinfeld and his collaborators during the 1980s. If one assumes compatibility with the compact form, the classification takes a particularly elegant form [2, 11, 49]: if $\mathfrak{g}$ is a complex simple Lie algebra and G is the simply connected compact Lie group corresponding to $\mathfrak{g}$, any infinitesimal deformation of the universal enveloping algebra $\mathscr{U}(\mathfrak{g})$ as a quasitriangular $*$-Hopf algebra corresponds to a Poisson–Lie group structure on G, and those are parametrized, up to isomorphisms and rescalings, by invariant Poisson structures on a maximal torus. Moreover, these deformations make perfect sense in Woronowicz's framework and give rise to strict deformation quantization of the corresponding Poisson structures in the sense of Rieffel [46].

Extending this classification from infinitesimal to analytical level, or in other words, going from formal to strict deformation quantization, is a nontrivial task. Since there is no analogue of Kontsevich's classification theorem in the analytical setting, we cannot expect such a clean result as in the formal case without imposing first an additional symmetry. One possible idea is to require the deformations to have the same combinatorial structure of representation theory, meaning the fusion rules and dimensions of representations, as in the classical case. This implies that we can take the coalgebra of matrix coefficients independent of the deformation parameter, and then try to find a suitable algebra structure on it. Moreover, if we want to preserve quasitriangularity, the Faddeev–Reshetikhin–Takhtadzhyan method [47] reduces the problem to finding R-matrices with certain algebraic symmetry. Still, carrying out such a classification directly is not an easy task, and this has been only worked out for the quantum groups with representation theory of SU(2) [57] and of SU(3) [38, 39].

Our approach to this problem consists of a combination of cohomological and analytical methods for tensor categories and related constructions for quantum groups. Suppose we want to classify all quantum groups which have the same fusion rules and dimensions of representations as a compact group G. In view of Woronowicz's Tannaka–Krein duality theorem, the problem can be divided into three parts:

– classification of rigid C*-tensor categories $\mathscr{C}$ with fusion rules of G;
– classification of monoidal autoequivalences of $\mathscr{C}$;
– classification of unitary fiber functors $\mathscr{C} \to \mathrm{Hilb}_\mathrm{f}$ inducing the classical dimension function on the representation ring of G.

We will mainly concentrate on the third problem, where the most recent advances are. Let us explain the strategy in more detail, in roughly the same order as we proceed from Sect. 4 to Sect. 6.

The first step is to establish a duality between the category of unital H-C*-algebras and that of $(\operatorname{Rep} H)$-module categories for compact quantum groups H [8, 28]. Given an H-C*-algebra B, which represents a noncommutative H-space X, we take the category of H-equivariant Hermitian vector bundles on X, that is, of finitely generated projective H-equivariant Hilbert B-modules. We can consider the tensor product of such modules and finite dimensional unitary representations of H, which leads to the structure of a $(\operatorname{Rep} H)$-module category. An analogue of the reconstruction procedure in Woronowicz's Tannaka–Krein theorem implies that B can be recovered from this module category and the distinguished object in it represented by B itself.

Further pursuing this duality, we have the following correspondence for subclasses of H-C*-algebras and $(\operatorname{Rep} H)$-module categories [33]. Among the H-C*-algebras we consider the braided-commutative Yetter–Drinfeld algebras, while among the module categories we take the C*-tensor categories with module structure induced by a tensor functor from $\operatorname{Rep} H$. A motivating example for this duality is the coideal of the function algebra coming from a quantum subgroup K of H, on one side, and the forgetful functor $\operatorname{Rep} H \to \operatorname{Rep} K$, on the other. In this formulation the quantum subgroup coideals precisely correspond to the factorizations $\operatorname{Rep} H \to \mathscr{C} \to \mathrm{Hilb}_{\mathrm{f}}$ of the canonical fiber functor on $\operatorname{Rep} H$ through some C*-tensor category $\mathscr{C}$.

One of our discoveries is that the noncommutative Poisson boundary of the discrete dual of H, and its counterpart on the categorical side, fits very well into this scheme [34]. The noncommutative Poisson boundary, modeled on the classical theory for discrete groups initiated by Furstenberg [14], was introduced by Izumi [16] to understand the lack of minimality for infinite tensor product actions of non-Kac quantum groups. This theory requires the operator algebraic framework in an essential way, and that is why we need to work with compact quantum groups instead of, for example, cosemisimple Hopf algebras. We find that the categorical Poisson boundary has a universality property for what we call *amenable tensor functors*. A concrete implication is that if G_q is the Drinfeld–Jimbo q-deformation of a compact connected semisimple Lie group G with a maximal torus T, then for $q \neq 1$ the forgetful functor $\operatorname{Rep} G_q \to \operatorname{Rep} T$ is a universal tensor functor defining the classical dimension function on $\operatorname{Rep} G$. This shows that the undeformed classical torus $T < G_q$ can be detected already at the categorical level. This also implies that the dimension-preserving fiber functors on $\operatorname{Rep} G$ are parametrized by the invariant Poisson structures on the torus, or the Poisson–Lie group structures of G, as expected from the case of infinitesimal deformations.

Returning to the first problem of classifying rigid C*-tensor categories with fusion rules of G, there is a simple class of examples of such categories in addition to $\operatorname{Rep} G_q$. We have a natural grading of $\operatorname{Rep} G_q$ by the Pontryagin dual of the center of G, or more intrinsically, by the chain group of $\operatorname{Rep} G_q$. Then any $\mathbb{T}$-valued 3-cocycle on the chain group defines an associator, and we obtain a twisted tensor category, which has the same fusion rules as G. Note that this is analogous to, but much easier than, the famous Knizhnik–Zamolodchikov associator constructed by

Drinfeld, which relates Rep G and Rep G_q. What we wrote in the previous paragraph about the categories Rep G_q applies equally well to these twisted categories and allows us to classify dimension-preserving fiber functors on them. In the case of SU(n), up to monoidal equivalence, these exhaust the tensor categories with fusion rules of SU(n) [18, 21]. It seems reasonable to expect that a similar result is true, or at least close to be true, for other simple Lie groups. Finally, the second problem of classifying autoequivalences of such categories has been essentially solved in [30, 31].

To summarize our results, for any compact connected simple Lie group G, we classify up to isomorphism all compact quantum groups with the same fusion rules and dimensions of representations as for G which, moreover, have representation categories monoidally equivalent to Rep G_q for $q \neq 1$ or to twists of such categories by 3-cocycles on the dual of the center of G, see Theorems 6.3 and 6.8. Whether such categories exhaust all categories with fusion rules of G beyond the case of SU(n), as well as whether one can say something precise for $q = 1$, are the main remaining open questions.

2 Monoidal Categories

Throughout the exposition we fix a universe and assume that all categories are *small* [23]. See [32] for the details on the following notions.

2.1 C-Categories*

A *C*-category* is a category $\mathscr{C}$ with morphism sets $\mathscr{C}(U,V)$ that are complex Banach spaces, endowed with a complex conjugate involution $\mathscr{C}(U,V) \to \mathscr{C}(V,U)$, $T \mapsto T^*$, satisfying the C*-identity

$$\|T^*T\| = \|T\|^2 = \|TT^*\|,$$

and such that the composition of morphisms is bilinear and $\|ST\| \leq \|S\|\,\|T\|$. Unless said otherwise, we always assume that $\mathscr{C}$ is closed under finite direct sums and subobjects. The latter means that any idempotent in the endomorphism ring $\mathscr{C}(X) = \mathscr{C}(X,X)$ comes from a direct summand of X. The existence of finite direct sums guarantees that, for any $T\colon X \to Y$, the morphism T^*T is positive as an element in the C*-algebra $\mathscr{C}(X)$, which otherwise we would have to add as an additional axiom.

An object is called *simple* if its endomorphism ring is isomorphic to $\mathbb{C}$, and a C*-category is said to be *semisimple* if any object is isomorphic to a finite direct sum of simple ones. We then denote the isomorphism classes of simple objects by $\operatorname{Irr}\mathscr{C}$ and assume that this is an at most countable set. Many results admit formulations which do not require this assumption, and can be proved by considering subcategories

generated by countable sets of simple objects, but we leave this matter to the interested reader.

A *unitary functor*, or a *C*-functor*, is a linear functor of C*-categories $\mathcal{F}: \mathcal{C} \to \mathcal{C}'$ satisfying $\mathcal{F}(T^*) = \mathcal{F}(T)^*$.

A few times we will need to perform the following operation: starting from a C*-category $\mathcal{C}$, we replace the morphisms sets by some larger multiplicative system $\mathcal{D}(X, Y)$ naturally containing the original $\mathcal{C}(X, Y)$. Then we perform the *idempotent completion* to construct a new category $\mathcal{D}$. That is, we regard the projections $p \in \mathcal{D}(X)$ as objects in the new category, and take $q\mathcal{D}(X, Y)p$ as the morphism set from the object represented by $p \in \mathcal{D}(X)$ to the one by $q \in \mathcal{D}(Y)$. Then the embeddings $\mathcal{C}(X, Y) \to \mathcal{D}(X, Y)$ can be considered as a C*-functor $\mathcal{C} \to \mathcal{D}$.

2.2 C*-Tensor Categories

A *C*-tensor category* is a C*-category endowed with a unitary bifunctor

$$\otimes: \mathcal{C} \times \mathcal{C} \to \mathcal{C},$$

a distinguished object $\mathbb{1} \in \mathcal{C}$, and natural unitary isomorphisms

$$\mathbb{1} \otimes U \cong U \cong U \otimes \mathbb{1}, \qquad \Phi(U, V, W): (U \otimes V) \otimes W \to U \otimes (V \otimes W)$$

satisfying certain compatibility conditions. If these isomorphisms can be taken to be the identity morphisms, then $\mathcal{C}$ is said to be *strict*.

We denote by Hilb_f a category of finite dimensional Hilbert spaces with a strict model of tensor product (see, for example, [48] or [32, p. 37] for concrete realizations).

A *unitary tensor functor*, or a *C*-tensor functor*, between two C*-tensor categories $\mathcal{C}$ and $\mathcal{C}'$ is given by a triple $(\mathcal{F}_0, \mathcal{F}, \mathcal{F}_2)$, where $\mathcal{F}$ is a C*-functor $\mathcal{C} \to \mathcal{C}'$, $\mathcal{F}_0$ is a unitary isomorphism $\mathbb{1}_{\mathcal{C}'} \to \mathcal{F}(\mathbb{1}_{\mathcal{C}})$, and $\mathcal{F}_2$ is a collection of natural unitary isomorphisms $\mathcal{F}(U) \otimes \mathcal{F}(V) \to \mathcal{F}(U \otimes V)$ compatible with the structure morphisms of $\mathcal{C}$ and $\mathcal{C}'$.

As a rule, we denote tensor functors by just one symbol $\mathcal{F}$. A tensor functor between strict C*-tensor categories is said to be *strict* if $\mathcal{F}_0$ and $\mathcal{F}_2$ are the identity morphisms. The composition of tensor functors $\mathcal{F}$ and $\mathcal{F}'$ is defined by taking the usual composition of C*-functors and setting $(\mathcal{F}\mathcal{F}')_2 = \mathcal{F}(\mathcal{F}'_2)\mathcal{F}_2$ and $\mathcal{F}\mathcal{F}' = \mathcal{F}(\mathcal{F}'_0)\mathcal{F}_0$. A *natural transformation* of tensor functors is a natural transformation in the usual sense which is also compatible with the isomorphisms $\mathcal{F}_2$ and $\mathcal{F}_0$.

When the C*-functor part of $\mathcal{F}$ is an equivalence of categories, $\mathcal{F}$ is said to be a *unitary monoidal equivalence*. Analogously to the case of ordinary functors, unitary monoidal equivalences can be inverted (up to a natural unitary monoidal isomorphism) as unitary tensor functors, so for any $\mathcal{F}$ as above there is another unitary monoidal equivalence $\mathcal{F}': \mathcal{C}' \to \mathcal{C}$ such that $\mathcal{F}\mathcal{F}'$ and $\mathcal{F}'\mathcal{F}$ are naturally

unitarily monoidally isomorphic to the identity functors of the respective categories. If the target and source categories are the same, such a tensor functor $\mathcal{F}$ is called an *autoequivalence*, and we denote by $\mathrm{Aut}^{\otimes}(\mathcal{C})$ the group of autoequivalences of $\mathcal{C}$ considered up to natural unitary monoidal isomorphisms. Let us also note that a version of Mac Lane's coherence theorem [23, Chapter XI] says that any C*-tensor category is unitarily monoidally equivalent to a strict one.

When $\mathcal{C}$ is a C*-tensor category and $U \in \mathcal{C}$, an object V is said to be a *dual object* of U if there are morphisms $R \in \mathcal{C}(\mathbb{1}, V \otimes U)$ and $\bar{R} \in \mathcal{C}(\mathbb{1}, U \otimes V)$ satisfying the conjugate equations

$$(\iota_V \otimes \bar{R}^*)\Phi(R \otimes \iota_V) = \iota_V, \qquad (\iota_U \otimes R^*)\Phi(\bar{R} \otimes \iota_U) = \iota_U,$$

where we assumed for simplicity that the unit is strict, so that $\mathbb{1} \otimes U = U \otimes \mathbb{1} = U$. If any object in $\mathcal{C}$ admits a dual, $\mathcal{C}$ is said to be *rigid* and we denote a choice of a dual of $U \in \mathcal{C}$ by $\bar{U}$.

Any rigid C*-tensor category with simple unit has finite dimensional morphism spaces and hence is automatically semisimple. The quantity

$$d^{\mathcal{C}}(U) = \min_{(R,\bar{R})} \|R\| \left\|\bar{R}\right\|,$$

where $(R, \bar{R})$ runs through the set of solutions of conjugate equations as above, is called the *intrinsic dimension* of U. We omit the superscript $\mathcal{C}$ when there is no danger of confusion. A solution $(R, \bar{R})$ of the conjugate equations for U is called *standard* if

$$\|R\| = \|\bar{R}\| = d(U)^{1/2}.$$

Solutions of the conjugate equations for U are unique up to the transformations

$$(R, \bar{R}) \mapsto ((T^* \otimes \iota)R, (\iota \otimes T^{-1})\bar{R}).$$

Furthermore, if $(R, \bar{R})$ is standard, then such a transformation defines a standard solution if and only if T is unitary.

In a rigid C*-tensor category $\mathcal{C}$ we often fix standard solutions $(R_U, \bar{R}_U)$ of the conjugate equations for every object U. Then $\mathcal{C}$ becomes *spherical* in the sense that one has the equality $R_U^*(\iota \otimes T)R_U = \bar{R}_U^*(T \otimes \iota)\bar{R}_U$ for any $T \in \mathcal{C}(U)$. The normalized linear functional

$$\mathrm{tr}_U(T) = \frac{1}{d(U)} R_U^*(\iota \otimes T)R_U = \frac{1}{d(U)} \bar{R}_U^*(T \otimes \iota)\bar{R}_U$$

is a tracial state on the finite dimensional C*-algebra $\mathcal{C}(U)$, called the *normalized categorical trace*. It is independent of the choice of a standard solution.

For any semisimple C*-tensor category we denote by $K^+(\mathcal{C})$ the Grothendieck semiring of $\mathcal{C}$. As an additive semigroup it is generated by the isomorphism classes $[U]$ of objects in $\mathcal{C}$ and satisfies the relations $[U \oplus V] = [U] + [V]$, so it is the free commutative semigroup with generators $[U] \in \operatorname{Irr}\mathcal{C}$. The product is defined by $[U]\,[V] = [U \otimes V]$.

Let us fix representatives $\{U_s\}_{s\in\operatorname{Irr}\mathcal{C}}$ of the isomorphism classes of simple objects of $\mathcal{C}$. We will often use the subindex s to denote various construction related to U_s. For every object $X \in \mathcal{C}$, denote by $\Gamma_X = (a^X_{st})_{s,t\in\operatorname{Irr}\mathcal{C}}$ the matrix describing the multiplication by $[X]$,

$$[X]\,[U_t] = \sum_s a^X_{st}[U_s],$$

so $a^X_{st} = \dim \mathcal{C}(U_s, X \otimes U_t)$. It is not difficult to see that $\|\Gamma_X\| \le d^{\mathcal{C}}(X)$. Moreover, the same is true for any *dimension function* d in place of $d^{\mathcal{C}}$, by which one means a unital semiring homomorphism $d\colon K^+(\mathcal{C}) \to [0, +\infty)$ such that $d([X]) = d([\bar{X}])$ for all X.

Another algebraic structure naturally associated with a semisimple C*-tensor category $\mathcal{C}$ is the *chain group* $\operatorname{Ch}(\mathcal{C})$. It is the group with generators g_s, $s \in \operatorname{Irr}\mathcal{C}$, satisfying the relations $g_r = g_s g_t$ whenever U_r embeds into $U_s \otimes U_t$. It is closely related to the notion of *grading* on a category. For a discrete group Γ, we say that $\mathcal{C}$ is *graded over* Γ if we are given full subcategories $(\mathcal{C}_g)_{g\in\Gamma}$ such that $\mathbb{1} \in \mathcal{C}_e$, any object of $\mathcal{C}$ is isomorphic in an essentially unique way to a finite direct sum $\oplus_g X_g$, with $X_g \in \mathcal{C}_g$, and $X \otimes Y$ is isomorphic to an object in $\mathcal{C}_{gh}$ if $X \in \mathcal{C}_g$ and $Y \in \mathcal{C}_h$. The chain group $\operatorname{Ch}(\mathcal{C})$ defines a grading on $\mathcal{C}$: for every $g \in \operatorname{Ch}(\mathcal{C})$, the subcategory $\mathcal{C}_g$ consists of direct sums of simple objects U_s such that $g_s = g$. Any other grading over a group Γ defines a canonical homomorphism $\operatorname{Ch}(\mathcal{C}) \to \Gamma$, so the chain group is a universal group over which $\mathcal{C}$ is graded.

3 Compact Quantum Groups

Compact quantum groups, or compact matrix pseudogroups as introduced by Woronowicz [56], are our main object of interest. We again follow the presentation of [32].

3.1 Tannaka–Krein Duality

A *compact quantum* group G is represented by a unital C*-algebra $C(G)$ equipped with a unital $*$-homomorphism $\Delta\colon C(G) \to C(G) \otimes C(G)$ satisfying

- coassociativity: $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$,
- cancellation properties: the linear spans of

$$(C(G) \otimes 1)\Delta(C(G)) \text{ and } (1 \otimes C(G))\Delta(C(G))$$

are dense in $C(G) \otimes C(G)$.

There is a unique state h on $C(G)$ satisfying $(h \otimes \iota)\Delta = h(\cdot)1$ (and/or $(\iota \otimes h)\Delta = h(\cdot)1$) called the *Haar state*. If h is faithful, $C(G)$ is called the *reduced function algebra of* G, or G is called a *reduced* compact quantum group, and we are mainly interested in such cases. By taking the image of $C(G)$ in the GNS representation of h, we can always work with a reduced model.

A *finite dimensional unitary representation* of G is a unitary element $U \in B(H_U) \otimes C(G)$ such that $(\iota \otimes \Delta)(U) = U_{12}U_{13}$, where H_U is a finite dimensional Hilbert space. The *intertwiners* between two representations U and V are the linear maps T from H_U to H_V satisfying $V(T \otimes 1) = (T \otimes 1)U$. The tensor product of two representations U and V is defined by $U_{13}V_{23}$ and denoted by $U \mathbin{⫪} V$. This way, the category $\operatorname{Rep} G$ of finite dimensional unitary representations with intertwiners as morphisms and with tensor product $⫪$ becomes a semisimple C*-tensor category.

When ω is in the (pre)dual $B(H_U)_* \cong \bar{H}_U \otimes H_U$, the element $(\omega \otimes \iota)(U) \in C(G)$ is called the matrix coefficient associated with ω. The dense $*$-subalgebra of $C(G)$ spanned by the matrix coefficients of finite dimensional representations is called the *regular algebra of* G, and is denoted by $\mathbb{C}[G]$. This space is closed under the coproduct, and becomes a Hopf $*$-algebra. Its antipode is characterized by

$$(\iota \otimes S)(U) = U^* \text{ for } U \in \operatorname{Rep} G.$$

Let us put $\mathscr{U}(G) = \mathbb{C}[G]^*$. This space has the structure of a $*$-algebra, defined by duality from the Hopf $*$-algebra $(\mathbb{C}[G], \Delta)$. Every finite dimensional unitary representation U of G defines a $*$-representation π_U of $\mathscr{U}(G)$ on H_U by $\pi_U(\omega) = (\iota \otimes \omega)(U)$. When convenient, we omit π_U and write $\omega\xi$ instead of $\pi_U(\omega)\xi$ for $\xi \in H_U$.

An important ingredient of the duality for unitary representations of G is the Woronowicz character $f_1 \in \mathscr{U}(G)$, which we denote by ρ. Namely, the element $\rho \in \mathscr{U}(G)$ is uniquely determined by the properties that it is positive, invertible, $\operatorname{Tr}(\pi_U(\rho)) = \operatorname{Tr}(\pi_U(\rho^{-1}))$, and

$$\bar{U} = (j(\rho)^{1/2} \otimes 1)(j \otimes \iota)(U^*)(j(\rho)^{-1/2} \otimes 1) \in B(\bar{H}_U) \otimes \mathbb{C}[G],$$

is unitary, where U is any finite dimensional unitary representation and j denotes the canonical $*$-anti-isomorphism $B(H_U) \cong B(\bar{H}_U)$ defined by $j(T)\bar{\xi} = \overline{T^*\xi}$. This element $\bar{U}$, which is again a unitary representation, is called the *conjugate representation* of U. The representation $\bar{U}$ is dual to U in the sense of Sect. 2.2, and

a convenient choice of solutions of the conjugate equations for U is given by

$$R_U(1) = \sum_i \bar{\xi}_i \otimes \rho^{-1/2}\xi_i \text{ and } \bar{R}_U(1) = \sum_i \rho^{1/2}\xi_i \otimes \bar{\xi}_i,$$

where $\{\xi_i\}_i$ is an orthonormal basis in H_U. This solution is standard. In particular, the intrinsic dimension function on $\operatorname{Rep} G$ coincides with the quantum dimension

$$\dim_q U = \operatorname{Tr}(\pi_U(\rho)).$$

Therefore $\operatorname{Rep} G$ is a rigid C*-tensor category. The forgetful functor $U \mapsto H_U$ defines a strict unitary tensor functor into Hilb_f called the *canonical fiber functor of G*. Woronowicz's Tannaka–Krein duality theorem recovers the $*$-Hopf algebra $\mathbb{C}[G]$ from these categorical data.

Theorem 3.1 ([58]) *Let $\mathscr{C}$ be a rigid C*-tensor category with simple unit, and $\mathscr{F}\colon \mathscr{C} \to \mathrm{Hilb}_\mathrm{f}$ be a unitary tensor functor. Then there exist a compact quantum group G, a unitary monoidal equivalence $\mathscr{E}\colon \operatorname{Rep} G \to \mathscr{C}$, and a natural unitary monoidal isomorphism from the canonical fiber functor of G to $\mathscr{F}\mathscr{E}$. Moreover, the Hopf $*$-algebra $\mathbb{C}[G]$ is determined uniquely up to isomorphism.*

The key idea is that if $\mathscr{C} \subset \mathrm{Hilb}_\mathrm{f}$ and $\mathscr{F}$ is the embedding functor, then, with representatives $\{U_s\}_s$ of $\operatorname{Irr}\mathscr{C}$ and $H_s = \mathscr{F}(U_s)$, the coalgebra

$$\bigoplus_{s \in \operatorname{Irr}\mathscr{C}} B(H_s)_* \cong \bigoplus_{s \in \operatorname{Irr}\mathscr{C}} \bar{H}_s \otimes H_s$$

admits an associative product induced by irreducible decompositions of tensor products, which makes it into a bialgebra, analogously to the description of the product of matrix coefficients for usual compact groups. Moreover, standard solutions of the conjugate equations determine the involution by the formula

$$\bar{H}_s \otimes H_s \ni \bar{\xi} \otimes \eta \mapsto \overline{(\iota \otimes \xi^*)R_s} \otimes (\eta^* \otimes \iota)\bar{R}_s \in \bar{H}_{\bar{s}} \otimes H_{\bar{s}}. \tag{3.1}$$

This way one obtains the $*$-bialgebra $\mathbb{C}[G]$.

The representation semiring of G is defined as $R^+(G) = K^+(\operatorname{Rep} G)$. A possible way of saying when a compact quantum group is a deformation of a genuine group is as follows.

Definition 3.2 Given a compact group H, we say that a compact quantum group G is of *H-type* if there exists a semiring isomorphism $R^+(G) \cong R^+(H)$ preserving the (classical) dimensions of representations.

This definition is essentially due to Woronowicz [58]. In [1] the Hopf $*$-algebras $\mathbb{C}[G]$ for compact quantum groups G of H-type are called *dimension-preserving R^+-deformations* of $\mathbb{C}[H]$.

Problem 3.3 (cf. [58]) Given a compact connected Lie group H, classify the compact quantum groups of H-type.

Our aim is to develop a general method to attack this problem, as outlined in Sect. 1.

3.2 Cohomology of the Discrete Dual

Deformation problems for compact quantum groups are controlled by a cohomology theory of the dual discrete quantum groups, which plays a central role in our considerations. We again refer the reader to [32] for a more thorough discussion.

The $*$-algebra $\mathscr{U}(G)$ can be identified with $\prod_{s\in\operatorname{Irr} G} B(H_s)$ (the algebraic direct product) using the correspondence

$$\mathbb{C}[G]^* \ni \omega \leftrightarrow (\pi_{U_s}(\omega))_s \in \prod_{s\in\operatorname{Irr} G} B(H_s).$$

More generally, we can consider the algebra

$$\mathscr{U}(G^k) = (\mathbb{C}[G]^{\otimes k})^* \cong \prod_{s_1,\dots,s_k\in\operatorname{Irr} G} B(H_{s_1}) \otimes \cdots \otimes B(H_{s_k}),$$

and interpret it as the space of (possibly unbounded) k-point functions on the "discrete dual" quantum group $\hat{G}$. This is, for example, how the discrete quantum groups are defined in [53]. If G is a genuine commutative compact group, this agrees with the usual notion of functions on the k-th power of the Pontryagin dual group $\hat{G}$.

The spaces $\mathscr{U}(G^k)$ can be regarded as the components of the standard complex for the group cohomology with multiplicative scalar coefficients as follows. We call the invertible elements of $\mathscr{U}(G^k)$ the *k-cochains* on $\hat{G}$. Given a k-cochain ω, we put

$$\partial^0(\omega) = 1 \otimes \omega, \quad \partial^{k+1}(\omega) = \omega \otimes 1,$$
$$\partial^j(\omega) = \hat{\Delta}_j(\omega) \quad (\hat{\Delta} \text{ applied to the } j\text{-th position}),$$

(which are all in $\mathscr{U}(G^{k+1})$), and call

$$\partial(\omega) = \partial^0(\omega)\partial^2(\omega)\cdots\partial^{2\lfloor\frac{k+1}{2}\rfloor}(\omega)\partial^1(\omega^{-1})\partial^3(\omega^{-1})\cdots\partial^{2\lfloor\frac{k}{2}\rfloor+1}(\omega^{-1})$$

the *coboundary* of ω. When $\partial(\omega) = 1$, ω is said to be a *k-cocycle*. We denote the set of k-cocycles by $Z^k(\hat{G}; \mathbb{C}^\times)$. Two k-cocycles χ and χ' are said to be *cohomologous* if

$$\chi' = \partial^0(\omega)\partial^2(\omega)\cdots\partial^{2\lfloor\frac{k}{2}\rfloor}(\omega)\chi\partial^1(\omega^{-1})\partial^3(\omega^{-1})\cdots\partial^{2\lfloor\frac{k-1}{2}\rfloor+1}(\omega^{-1})$$

holds for some $(k-1)$-cochain ω.

In general the relation of being cohomologous is not even symmetric. But if it happens to be an equivalence relation, the set of equivalence classes of k-cocycles is denoted by $H^k(\hat{G}; \mathbb{C}^\times)$ and called the *k-cohomology* of $\hat{G}$. When one requires all the ingredients to be unitary instead of invertible, the corresponding set is denoted by $H^k(\hat{G}; \mathbb{T})$. If a k-cochain ω commutes with the image of $\hat{\Delta}^{k-1}$ (defined inductively as $(\hat{\Delta}^{k-2} \otimes \iota)\hat{\Delta}$), we say that ω is *invariant*. Invariant cocycles are also called *lazy* in the algebraic literature. If we consider only invariant cocycles (with coboundaries defined also using only invariant cochains), we get sets $H^k_G(\hat{G}; \mathbb{C}^\times)$, and $H^k_G(\hat{G}; \mathbb{T})$ for the unitary case, whenever these sets are well-defined.

When G is a commutative group, $H^k_G(\hat{G}; \mathbb{T}) = H^k(\hat{G}; \mathbb{T})$ agrees with the usual group cohomology of the Pontryagin dual. We also note that, in general, if H is a quantum subgroup of G, the natural inclusion $\mathscr{U}(H^k) \to \mathscr{U}(G^k)$ induces maps from the cohomologies of $\hat{H}$ to those of $\hat{G}$.

We are mainly interested in the cohomology in low degrees ($k \leq 3$), which have direct connections with various aspects of $\operatorname{Rep} G$. Let us briefly summarize these connections.

The set $H^1(\hat{G}; \mathbb{T}) = Z^1(\hat{G}; \mathbb{T})$ consists of unitary group-like elements in $\mathscr{U}(G)$. If G is a genuine compact group, then any such group-like element arises from an element of G, so we have canonical isomorphisms

$$H^1(\hat{G}; \mathbb{T}) \cong G \quad \text{and} \quad H^1_G(\hat{G}; \mathbb{T}) \cong Z(G).$$

For arbitrary G, a categorical interpretation of $H^1(\hat{G}; \mathbb{T})$ is that this is the group of unitary monoidal automorphisms of the canonical fiber functor $\mathscr{F}\colon \operatorname{Rep} G \to \mathrm{Hilb}_\mathrm{f}$, while the group $H^1_G(\hat{G}; \mathbb{T})$ is the group of unitary monoidal automorphisms of the identity functor on $\operatorname{Rep} G$.

Next, a unitary 2-cochain $x \in \mathscr{U}(G^2)$ is a 2-cocycle if and only if it satisfies

$$(x \otimes 1)(\hat{\Delta} \otimes \iota)(x) = (1 \otimes x)(\iota \otimes \hat{\Delta})(x).$$

If c is a unitary element in the center of $\mathscr{U}(G)$ (an invariant unitary 1-cochain), then its coboundary is $(c \otimes c)\hat{\Delta}(c^{-1})$. The set of invariant unitary 2-cocycles forms a group under multiplication, and the coboundaries form a subgroup. Thus, $H^2_G(\hat{G}; \mathbb{T})$ becomes a group, called the *invariant* 2*-cohomology group* of $\hat{G}$.

If x is an invariant unitary 2-cocycle, the multiplication by x^{-1} on $H_U \otimes H_V$ can be considered as a unitary endomorphism of $U \mathbin{\bigcirc\!\!\!\!\!\top} V$ in $\operatorname{Rep} G$. Such endomorphisms form a natural unitary transformation of the bifunctor $\mathbin{\bigcirc\!\!\!\!\!\top}$ into itself. The cocycle condition corresponds to the fact that this transformation is a monoidal autoequivalence of $\operatorname{Rep} G$. Up to natural unitary monoidal isomorphisms, any autoequivalence of $\operatorname{Rep} G$ fixing the irreducible classes can be obtained in this way. Moreover, the cohomology relation of cocycles corresponds to the natural unitary monoidal isomorphism of autoequivalences. Thus $H^2_G(\hat{G}; \mathbb{T})$ can be considered as the normal subgroup of $\operatorname{Aut}^\otimes(\operatorname{Rep} G)$ consisting of (isomorphism classes of) autoequivalences that preserve the isomorphism classes of objects. Without the unitarity, $H^2_G(\hat{G}; \mathbb{C}^\times)$

corresponds to a subgroup of monoidal autoequivalences of $\operatorname{Rep} G$ as a tensor category over $\mathbb{C}$.

Let x be an arbitrary unitary 2-cocycle on $\hat{G}$, invariant or not. Then the triple $\mathscr{F}_x = (\mathrm{id}_{\mathbb{C}}, U \mapsto H_U, x^{-1})$ defines a new unitary tensor functor $\operatorname{Rep} G \to \mathrm{Hilb}_{\mathrm{f}}$. By Theorem 3.1, $\mathscr{F}_x$ can be considered as the canonical fiber functor of another compact quantum group G_x satisfying $\operatorname{Rep} G = \operatorname{Rep} G_x$. Concretely, $\mathscr{U}(G_x)$ coincides with $\mathscr{U}(G)$ as a $*$-algebra, but is endowed with the modified coproduct $\hat{\Delta}_x(T) = x\hat{\Delta}(T)x^{-1}$. By duality, $\mathbb{C}[G_x]$ is the same coalgebra as $\mathbb{C}[G]$, but has a modified $*$-algebra structure (pre)dual to $(\mathscr{U}(G), \hat{\Delta}_x)$.

Up to natural unitary monoidal isomorphisms, the functors $\mathscr{F}_x$ exhaust all unitary tensor functors $\mathscr{F}'\colon \operatorname{Rep} G \to \mathrm{Hilb}_{\mathrm{f}}$ satisfying $\dim \mathscr{F}'(U) = \dim H_U$. Moreover, if T is a unitary element in $\mathscr{U}(G)$, then $x_T = (T \otimes T)x\hat{\Delta}(T^{-1})$ defines another unitary fiber functor which is naturally unitarily monoidally isomorphic to $\mathscr{F}_x$. Therefore the set $H^2(\hat{G}; \mathbb{T})$ gives a parametrization of the set of isomorphism classes of dimension-preserving unitary fiber functors $\operatorname{Rep} G \to \mathrm{Hilb}_{\mathrm{f}}$.

The group $H^2_G(\hat{G}; \mathbb{T})$ acts on the set $H^2(\hat{G}; \mathbb{T})$ by multiplication on the right. This corresponds to the restriction of the obvious right action of $\mathrm{Aut}^{\otimes}(\operatorname{Rep} G)$ on the natural unitary monoidal isomorphism classes of unitary fiber functors $\operatorname{Rep} G \to \mathrm{Hilb}_{\mathrm{f}}$. Note that by Theorem 3.1, the orbits of this action by $\mathrm{Aut}^{\otimes}(\operatorname{Rep} G)$ precisely correspond to the isomorphism classes of compact quantum groups with representation category $\operatorname{Rep} G$.

Let us move on to 3-cohomology. An invertible element $\Phi \in \mathscr{U}(G^3)$ is a 3-cocycle if and only if it satisfies

$$(1 \otimes \Phi)(\iota \otimes \hat{\Delta} \otimes \iota)(\Phi)(\Phi \otimes 1) = (\iota \otimes \iota \otimes \hat{\Delta})(\Phi)(\hat{\Delta} \otimes \iota \otimes \iota)(\Phi).$$

Invariant 3-cocycles are also called *associators*. If Φ is such a cocycle, its action on $H_U \otimes H_V \otimes H_W$ can be considered as a new associativity morphism on the C*-category $\operatorname{Rep} G$ with bifunctor $\mathbin{\bigcirc\!\!\!\!\!\top}$. If Φ is unitary, this gives a new C*-tensor category $(\operatorname{Rep} G, \Phi)$, which has the same data as $\operatorname{Rep} G$ except for the new associativity morphisms defined by the action of Φ. If $\mathscr{C}$ is a semisimple C*-tensor category with the same fusion rules as $\operatorname{Rep} G$, that is, with the same Grothendieck semiring, then by transporting the monoidal structure of $\mathscr{C}$ along any choice of a C*-functor $\mathscr{C} \to \operatorname{Rep} G$ defining the isomorphism $K^+(\mathscr{C}) \cong R^+(G)$, we see that $\mathscr{C}$ is monoidally equivalent to some $(\operatorname{Rep} G, \Phi)$ as above. We remark that in general it is not clear whether any such category is automatically rigid.

If x is an invariant unitary 2-cochain, the categories $(\operatorname{Rep} G, \Phi)$ and $(\operatorname{Rep} G, \Phi_x)$ are naturally unitarily monoidally equivalent, by means of the unitary tensor functor $(\mathrm{id}_{\mathbb{1}}, \mathrm{Id}_{\operatorname{Rep} G}, x^{-1})\colon (\operatorname{Rep} G, \Phi) \to (\operatorname{Rep} G, \Phi_x)$. This way the set $H^3_G(\hat{G}; \mathbb{T})$ gives a parametrization of the categories of the form $(\operatorname{Rep} G, \Phi)$ considered up to unitary monoidal equivalences that preserve the isomorphism classes of objects.

A simple way of constructing elements of $H^3_G(\hat{G}; \mathbb{T})$ is by considering the chain group $\mathrm{Ch}(\operatorname{Rep} G)$, which we denote by $\mathrm{Ch}(G)$. Namely, any cocycle $\phi \in Z^3(\mathrm{Ch}(\mathscr{C}); \mathbb{T})$ can be considered as a 3-cocycle on $\hat{G}$ such that its component in $B(H_r) \otimes B(H_s) \otimes B(H_t)$ is the scalar $\phi(g_r, g_s, g_t)$. We denote the category $\operatorname{Rep} G$

with the associator given by this cocycle by $(\operatorname{Rep} G)^\phi$. Therefore in $(\operatorname{Rep} G)^\phi$ the associativity morphism $(U \mathbin{\textcircled{\scriptsize T}} V) \mathbin{\textcircled{\scriptsize T}} W \to U \mathbin{\textcircled{\scriptsize T}} (V \mathbin{\textcircled{\scriptsize T}} W)$ is the scalar operator $\phi(g,h,k)$ if U, V and W have the degrees g, h and $k \in \operatorname{Ch}(G)$, respectively. The category $(\operatorname{Rep} G)^\phi$ is always rigid.

Let us note in passing that the Pontryagin dual of the abelianization $\operatorname{Ch}(G)^{\mathrm{ab}}$ of the chain group is naturally isomorphic to $H^1_G(\hat{G};\mathbb{T})$, cf. [4, 27]. Namely, any character $\chi\colon \operatorname{Ch}(G) \to \mathbb{T}$ defines a 1-cocycle $(\chi(g_s))_{s\in \operatorname{Irr}\mathscr{C}} \in \mathscr{U}(G)$, or equivalently, a monoidal automorphism η^χ of the identity functor on $\operatorname{Rep} G$ such that $\eta^\chi_U = \chi(g)\iota_U$ if U has degree $g \in \operatorname{Ch}(G)$.

In particular, if G is a genuine group, then $\operatorname{Ch}(G)$ is abelian and we get an isomorphism $\operatorname{Ch}(G) \cong \widehat{Z(G)}$. More concretely, if U is an irreducible representation of G, then the group $Z(G)$ must be acting by scalars, that is, by some character $\chi_U \in \widehat{Z(G)}$, and the isomorphism $\operatorname{Ch}(G) \cong \widehat{Z(G)}$ maps g_s into χ_{U_s}.

3.3 q-Deformation

Let G be a simply connected compact Lie group with a maximal torus T, $\mathfrak{g}$ be the corresponding semisimple complex Lie algebra. Fix a system of simple roots and denote the Cartan matrix by $(a_{ij})_{i,j}$. We fix a scalar product defining the root system of G. When G is simple, such a scalar product is unique up to a scalar factor and we normalize it so that for every short root α_i we have $(\alpha_i,\alpha_i) = 2$.

For a fixed positive real number $q \neq 1$, the quantized universal enveloping algebra $\mathscr{U}_q(\mathfrak{g})$ is the $*$-Hopf algebra (over $\mathbb{C}$) generated by elements $E_i, F_i, K_i^{\pm 1}$ satisfying

$$[K_i,K_j] = 0, \quad K_iE_jK_i^{-1} = q_i^{a_{ij}}E_j, \quad K_iF_jK_i^{-1} = q_i^{-a_{ij}}F_j, \quad [E_i,F_j] = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} E_i^k E_j E_i^{1-a_{ij}-k} = 0,$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} F_i^k F_j F_i^{1-a_{ij}-k} = 0.$$

where $q_i = q^{d_i}$ and $d_i = (\alpha_i,\alpha_i)/2$. The coproduct is

$$\hat{\Delta}_q(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \hat{\Delta}_q(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \hat{\Delta}_q(K_i) = K_i \otimes K_i,$$

and the involution is characterized by

$$E_i^* = F_iK_i, \quad F_i^* = K_i^{-1}E_i, \quad K_i^* = K_i.$$

A finite dimensional representation of $\mathscr{U}_q(\mathfrak{g})$ is said to be *admissible* if it defines a representation of the maximal torus T of G, so that the elements K_i act as the elements $q_i^{H_i}$ in the complexification of T, where H_i is the element of the Cartan subalgebra of $\mathfrak{g}$ defined by $\alpha_j(H_i) = a_{ij}$. The finite dimensional admissible unitary representations form a rigid C*-tensor category. Since it is given as a subcategory of Hilb_f, Woronowicz's Tannaka–Krein theorem gives a compact quantum group, which is called the *q-deformation* of G_q. The application of Woronowicz's theorem is a bit of an overkill here, and concretely, the Hopf $*$-algebra $\mathbb{C}[G_q]$ is defined as the subalgebra of $\mathscr{U}_q(\mathfrak{g})^*$ generated by the matrix coefficients of admissible representations. It is known that the quantum group G_q is of G-type according to Definition 3.2.

The algebra of the discrete dual $\mathscr{U}(G_q)$ can be regarded as a completion of $\mathscr{U}_q(\mathfrak{g})$ (although one should note that the latter has non-admissible representations which do not extend to $\mathscr{U}(G_q)$). The quantum group G_q contains T as a closed subgroup, with the corresponding embedding $\mathscr{U}(T) \hookrightarrow \mathscr{U}(G_q)$ given by identifying $q_i^{H_i}$ with K_i. Intrinsically the torus T can be characterized as the *maximal quantum Kac subgroup* of G_q, that is, the largest quantum subgroup of G_q on which the antipode is involutive [51].

The center $Z(G) \subset T$ of G is contained in the center of $\mathscr{U}(G_q)$. It follows that for any subgroup $\Gamma \subset Z(G)$ the quantum group G_q/Γ makes sense. This allows one to deform compact connected semisimple Lie groups that are not necessarily simply connected by letting $(G/\Gamma)_q = G_q/\Gamma$. In other words, $(G/\Gamma)_q$ is obtained from $\mathscr{U}_q(\mathfrak{g})$ by considering only the admissible representations such that their restrictions to T factor through T/Γ, i.e., that have weights annihilating Γ.

It is known that for $q \neq 1$ we have $H^1(\hat{G}_q; \mathbb{T}) = T$ and $H^1_{G_q}(\hat{G}_q; \mathbb{T}) = Z(G)$, which is a simple consequence of Soĭbelman's classification of irreducible representations of $\mathbb{C}[G_q]$ [50]. The second cohomology will be described in Sects. 5.2 and 6.1. As for the third cohomology, since G_q and G are known to have the same fusion rules, there is a canonical bijection $H^3_{G_q}(\hat{G}_q; \mathbb{T}) \cong H^3_G(\hat{G}; \mathbb{T})$. For the same reason there is a unique element in $H^3_G(\hat{G}; \mathbb{T})$ corresponding to $\operatorname{Rep} G_q$. An associator $\Phi_{\mathrm{KZ},q}$ representing this element was constructed by Drinfeld [11] using the Knizhnik–Zamolodchikov equations (see also [20, 29]). Since the categories $\operatorname{Rep} G_q$ are mutually inequivalent for different $q \in (0, 1]$, we thus get a family of different classes $[\Phi_{\mathrm{KZ},q}] \in H^3_G(\hat{G}; \mathbb{T})$ indexed by $q \in (0, 1]$. By our discussion in Sect. 3.2, there also are classes in $H^3_G(\hat{G}; \mathbb{T})$ coming from 3-cocycles on $\mathrm{Ch}(G) \cong \widehat{Z(G)}$, as well as from their products with $\Phi_{\mathrm{KZ},q}$. What else is contained in $H^3_G(\hat{G}; \mathbb{T})$, is a major open problem.

4 Categorical Duality for Actions of Quantum Groups

In this section we describe several extensions of the Tannaka–Krein duality to actions of compact quantum groups. These extensions are not strictly speaking needed for the classification problem for compact quantum groups, but they provide motivation for some of the subsequent constructions.

4.1 G-Algebras and (Rep G)*-Module Categories*

Given a compact quantum group G, a unital G-C*-algebra is a unital C*-algebra B equipped with a continuous left action $\alpha\colon B \to C(G) \otimes B$ of G. This means that α is an injective unital $*$-homomorphism such that $(\Delta \otimes \iota)\alpha = (\iota \otimes \alpha)\alpha$ and such that the space $(C(G) \otimes 1)\alpha(B)$ is dense in $C(G) \otimes B$. The linear span of spectral subspaces,

$$\mathscr{B} = \{x \in B \mid \alpha(x) \in \mathbb{C}[G] \otimes_{\mathrm{alg}} B\},$$

which is a dense $*$-subalgebra of B, is called the *regular subalgebra* of B, and the elements of $\mathscr{B}$ are called *regular*. More concretely, the algebra $\mathscr{B}$ is spanned by the elements of the form $(h \otimes \iota)((x \otimes 1)\alpha(a))$ for $x \in \mathbb{C}[G]$ and $a \in B$. This algebra is of central importance for the categorical reconstruction of B.

Next let us explain the categorical counterpart of the G-algebras. We also note that a particular type of this structure plays a central role in the subfactor theory.

Definition 4.1 Let $\mathscr{C}$ be a C*-tensor category. We say that $\mathscr{D}$ is *a right $\mathscr{C}$-module category* if

- $\mathscr{D}$ is a C*-category,
- we are given a unitary bifunctor $\mathscr{D} \times \mathscr{C} \to \mathscr{D}$, denoted by $(X, U) \mapsto X \times U$, and
- natural unitary isomorphisms

$$X \times \mathbb{1} \to X, \quad (X \times U) \times V \to X \times (U \otimes V) \tag{4.1}$$

satisfying the obvious compatibility conditions analogous to those for monoidal categories.

When $\mathscr{D}$ and $\mathscr{D}'$ are $\mathscr{C}$-module categories, a *$\mathscr{C}$-module functor $\mathscr{D} \to \mathscr{D}'$* is a pair $(\mathscr{F}, \theta)$, where $\mathscr{F}$ is a C*-functor from $\mathscr{D}$ to $\mathscr{D}'$, and θ is a unitary natural transformation $\mathscr{F}(X) \times U \to \mathscr{F}(X \times U)$, again satisfying a standard set of compatibility conditions. When θ is obvious from context, we simply write $\mathscr{F}$ instead of $(\mathscr{F}, \theta)$. The $\mathscr{C}$-module functors can be composed in a way similar to the monoidal functors.

We say that two $\mathscr{C}$-module functors $(\mathscr{F}, \theta), (\mathscr{F}', \theta')\colon \mathscr{D} \to \mathscr{D}'$ are equivalent if there is a natural unitary transformation $\eta\colon \mathscr{F} \to \mathscr{F}'$ which is compatible with θ

and θ'. Note that this equivalence relation is compatible with composition of module functors.

Our starting point is the following categorical duality theorem for G-algebras, which could be called the Tannaka–Krein duality theorem for quantum group actions. Results leading to this theorem have a long history, starting from the work of Wassermann [54] and Landstad [22] in the early 1980s on full multiplicity ergodic actions of compact groups, and continuing in [3, 40, 43, 52].

Theorem 4.2 ([8, 28]) *Let G be a reduced compact quantum group. Then the following two categories are equivalent:*

1. *The category of unital G-C^*-algebras, with unital G-equivariant $*$-homomorphisms as morphisms.*
2. *The category of pairs $(\mathcal{D}, M)$, where $\mathcal{D}$ is a right* $(\operatorname{Rep} G)$*-module C^*-category and M is a generating object in $\mathcal{D}$, with equivalence classes of unitary* $(\operatorname{Rep} G)$*-module functors respecting the prescribed generating objects as morphisms.*

The condition that $\mathcal{D}$ is generated by M means that any object in $\mathcal{D}$ is isomorphic to a subobject of $M \times U$ for some $U \in \operatorname{Rep} G$.

Remark 4.3 Let $\operatorname{End}(\mathcal{D})$ be the C*-tensor category of C*-endofunctors on $\mathcal{D}$, with uniformly bounded natural transformations as morphisms. Then, having a $\mathcal{C}$-module structure on $\mathcal{D}$ is the same as giving a unitary tensor functor $\mathcal{F}\colon \mathcal{C}^{\otimes \mathrm{op}} \to \operatorname{End}(\mathcal{D})$, where $\mathcal{F}_0$ and $\mathcal{F}_2$ correspond to the morphisms in (4.1). For $G = \mathrm{SU}_q(2)$ and, more generally, for free orthogonal quantum groups this point of view leads to a combinatorial classification of ergodic actions [9].

To describe the above equivalence, given a G-C*-algebra (B, α), we consider the category $\mathcal{D}_B$ of G-equivariant finitely generated right Hilbert B-modules. In other words, objects of $\mathcal{D}_B$ are finitely generated right Hilbert B-modules X equipped with a linear map $\delta = \delta_X\colon X \to C(G) \otimes X$ which satisfies the comultiplicativity property $(\Delta \otimes \iota)\delta = (\iota \otimes \delta)\delta$ together with the following conditions:

- $(C(G) \otimes 1)\delta(X)$ is dense in $C(G) \otimes X$,
- δ is compatible with the Hilbert B-module structure, in the sense that

$$\delta(\xi a) = \delta(\xi)\alpha(a), \quad \langle \delta(\xi), \delta(\zeta) \rangle = \alpha(\langle \xi, \zeta \rangle), \quad (\xi, \zeta \in X, a \in B).$$

Here, $C(G) \otimes X$ is considered as a right Hilbert $(C(G) \otimes B)$-module.

For $X \in \mathcal{D}_B$ and $U \in \operatorname{Rep} G$, we obtain a new object $X \times U$ in $\mathcal{D}_B$ given by the linear space $H_U \otimes X$, which is a right Hilbert B-module such that

$$(\xi \otimes x)a = \xi \otimes xa, \quad \langle \xi \otimes x, \eta \otimes y \rangle_B = (\eta, \xi)\langle x, y \rangle_B \quad \text{for } \xi, \eta \in H_U,\ x, y \in X,\ a \in B,$$

together with the compatible $C(G)$-coaction map

$$\delta = \delta_{H_U \otimes X}\colon H_U \otimes X \to C(G) \otimes H_U \otimes X, \quad \delta(\xi \otimes x) = U_{21}^*(\xi \otimes \delta_X(x))_{213}.$$

This construction gives the structure of a right $(\operatorname{Rep} G)$-module category on $\mathscr{D}_B$, together with a distinguished object $B \in \mathscr{D}_B$.

In the other direction, we construct what would be the regular subalgebra of a G-algebra starting from a pair $(\mathscr{D}, M)$ as in Theorem 4.2. The generating condition on M implies that, by replacing $\mathscr{D}$ by an equivalent category, we may assume that $\mathscr{D}$ is the idempotent completion of the category $\operatorname{Rep} G$ with larger morphism sets $\mathscr{D}(U, V)$ than in $\operatorname{Rep} G$, such that M is the unit object $\mathbb{1}$ in $\operatorname{Rep} G$ and the functor $\iota \times U$ on $\mathscr{D}$ is an extension of the functor $\iota \mathbin{\bigcirc\!\!\!\!\!\top} U$ on $\operatorname{Rep} G$. Namely, we simply define the new set of morphisms between U and V as $\mathscr{D}(M \times U, M \times V)$.

Consider the linear space

$$\mathscr{B} = \bigoplus_{s \in \operatorname{Irr} G} \bar{H}_s \otimes \mathscr{D}(\mathbb{1}, U_s).$$

Note that this gives $\mathbb{C}[G]$ if $\mathscr{D}$ is Hilb_f with the action of $\operatorname{Rep} G$ induced by the canonical fiber functor. Using this observation, the associative product on $\mathscr{B}$ is defined by

$$(\bar{\xi} \otimes T) \cdot (\bar{\zeta} \otimes S) = \sum_{r \in \operatorname{Irr} G} \overline{u_{ts}^{r\alpha *}(\xi \otimes \zeta)} \otimes u_{ts}^{r\alpha *}(T \otimes S),$$

where $(u_{ts}^{r\alpha})_\alpha$ is an orthonormal basis in the space of morphisms $U_r \to U_t \mathbin{\bigcirc\!\!\!\!\!\top} U_s$. Similarly, the $*$-structure on $\mathscr{B}$ is given by the following analogue of (3.1):

$$(\bar{\xi} \otimes T)^* = \rho^{-1/2} \xi \otimes (T^* \otimes \iota)\bar{R}_t \quad (\bar{\xi} \otimes T \in \bar{H}_t \otimes \mathscr{D}(\mathbb{1}, U_t)).$$

The $*$-algebra $\mathscr{B}$ has a natural left $\mathbb{C}[G]$-comodule structure, defined by the map $\alpha\colon \mathscr{B} \to \mathbb{C}[G] \otimes \mathscr{B}$ such that if $\{\xi_i\}_i$ is an orthonormal basis in H_t and u_{ij} are the matrix coefficients of U_t in this basis, then

$$\alpha(\bar{\xi}_i \otimes T) = \sum_j u_{ij} \otimes \bar{\xi}_j \otimes T.$$

It is shown then that the action α is algebraic in the sense of [8, Definition 4.2], meaning that the fixed point algebra $A = \mathscr{B}^G \cong \mathscr{D}(\mathbb{1})$ is a unital C*-algebra and the conditional expectation $(h \otimes \iota)\alpha\colon \mathscr{B} \to A$ is positive and faithful. It follows that there is a unique completion of $\mathscr{B}$ to a C*-algebra B such that α extends to an action of the reduced form of G on B. This finishes the construction of an action from a module category.

4.2 Yetter–Drinfeld Algebras and Tensor Functors

An important class of G-C*-algebras is the quantum homogeneous spaces defined by quantum subgroups. Namely, when $f\colon \mathbb{C}[G] \to \mathbb{C}[H]$ is a surjective homomorphism of Hopf $*$-algebras, the subalgebra

$$\mathbb{C}[G/H] = \{x \in \mathbb{C}[G] \colon (\iota \otimes f)\Delta(x) = x \otimes 1\}$$

completes to a $C(G)$-comodule subalgebra of $C(G)$. As the notation suggests, this should be regarded as the space of functions on the quotient G/H. The categorical counterpart to this structure is the $(\operatorname{Rep} G)$-module category $\operatorname{Rep} H$, where $\operatorname{Rep} G$ acts through the unitary tensor functor $\operatorname{Rep} G \to \operatorname{Rep} H$ induced by f.

In general, we could ask when a $(\operatorname{Rep} G)$-module category structure comes from a tensor functor, or in other words, when the G-equivariant Hilbert B-modules admit a tensor product operation. The answer is that the G-algebra structure of B should be extended to that of a *braided-commutative Yetter–Drinfeld G-algebra*. Let us briefly recall the relevant definitions.

Assume we have a continuous left action of $\alpha\colon B \to C(G) \otimes B$ of a compact quantum group G on a unital C*-algebra B, as well as a continuous right action $\beta\colon B \to \mathscr{M}(B \otimes c_0(\hat{G}))$ of the dual discrete quantum group $\hat{G}$. The action β defines a left $\mathbb{C}[G]$-module algebra structure $\rhd\colon \mathbb{C}[G] \otimes B \to B$ on B by

$$x \rhd a = (\iota \otimes x)\beta(x) \ \text{ for } \ x \in \mathbb{C}[G] \ \text{ and } \ a \in B.$$

Here we view $c_0(\hat{G})$ as a subalgebra of $\mathscr{U}(G) = \mathbb{C}[G]^*$. This structure is compatible with involution, in the sense that

$$x \rhd a^* = (S(x)^* \rhd a)^*.$$

Definition 4.4 We say that B is a *Yetter–Drinfeld G-C*-algebra* if the following identity holds for all $x \in \mathbb{C}[G]$ and $a \in \mathscr{B}$:

$$\alpha(x \rhd a) = x_{(1)}a_{(1)}S(x_{(3)}) \otimes (x_{(2)} \rhd a_{(2)}),$$

where we use Sweedler's sumless notation, so we write the effect of Δ and α as $x_{(1)} \otimes x_{(2)}$, etc. A Yetter–Drinfeld G-C*-algebra B is said to be *braided-commutative* if for all $a, b \in \mathscr{B}$ we have

$$ab = b_{(2)}(S^{-1}(b_{(1)}) \rhd a). \tag{4.2}$$

Note that when b is in the fixed point algebra $A = \mathscr{B}^G$, the right hand side of the above identity reduces to ba, and we see that A is contained in the center of $\mathscr{B}$.

Remark 4.5 Yetter–Drinfeld G-C*-algebras can be regarded as $D(G)$-C*-algebras for the Drinfeld double $D(G)$ of G, and they are studied in the more general setting of locally compact quantum groups by Nest and Voigt [37].

The categorical counterpart of a braided-commutative Yetter–Drinfeld G-C*-algebra is a pair $(\mathscr{C},\mathscr{E})$, where $\mathscr{C}$ is a C*-tensor category and $\mathscr{E}\colon \operatorname{Rep} G \to \mathscr{C}$ is a unitary tensor functor such that $\mathscr{C}$ is generated by the image of $\mathscr{E}$. The condition that $\mathscr{C}$ is generated by the image of $\mathscr{E}$ means that any object in $\mathscr{C}$ is isomorphic to a subobject of $\mathscr{E}(U)$ for some $U \in \operatorname{Rep} G$. We stress that we do not assume that the unit in $\mathscr{C}$ is simple. In fact, the C*-algebra $\mathscr{C}(\mathbb{1})$ is exactly the fixed point subalgebra B^G of the C*-algebra B corresponding to $(\mathscr{C},\mathscr{E})$ in the next theorem.

Define the morphisms $(\mathscr{C},\mathscr{E}) \to (\mathscr{C}',\mathscr{E}')$ as the equivalence classes of pairs $(\mathscr{F},\eta)$, where $\mathscr{F}$ is a unitary tensor functor $\mathscr{F}\colon\mathscr{C} \to \mathscr{C}'$ and η is a natural unitary monoidal isomorphism $\eta\colon\mathscr{F}\mathscr{E} \to \mathscr{E}'$. We say that $(\mathscr{F},\eta)$ and $(\mathscr{F}',\eta')$ are equivalent if there is a natural unitary monoidal transformation of the unitary tensor functors $\xi\colon\mathscr{F} \to \mathscr{F}'$ which is compatible with η and η' in the sense that $\eta_U = \eta'_U\xi_{\mathscr{E}(U)}\colon\mathscr{F}(\mathscr{E}(U)) \to \mathscr{E}'(U)$ for all $U \in \mathscr{C}$. Again, this relation is compatible with the composition of morphisms, and we denote by $\mathcal{T}\!ens(\operatorname{Rep} G)$ the category of pairs $(\mathscr{C},\mathscr{E})$ with morphisms given by the equivalence classes of this relation.

Theorem 4.6 ([33]) *Let G be a reduced compact quantum group. Then the following two categories are equivalent:*

1. *The category $\mathcal{YD}_{\mathrm{brc}}(G)$ of unital braided-commutative Yetter–Drinfeld G-C*-algebras, with unital G- and $\hat{G}$-equivariant $*$-homomorphisms as morphisms.*
2. *The category $\mathcal{T}\!ens(\operatorname{Rep} G)$.*

Moreover, given a morphism $[(\mathscr{F},\eta)]\colon(\mathscr{C},\mathscr{E}) \to (\mathscr{C}',\mathscr{E}')$, the corresponding homomorphism of Yetter–Drinfeld C*-algebras is injective if and only if $\mathscr{F}$ is faithful, and it is surjective if and only if $\mathscr{F}$ is full.

To prove this theorem we have to enhance the ingredients of Theorem 4.2. Suppose that B is a Yetter–Drinfeld G-C*-algebra. Then any G-equivariant right Hilbert B-module X automatically admits a left action of $\mathscr{B}$, given by

$$a\xi = \xi_{(2)}(S^{-1}(\xi_{(1)}) \rhd a) \quad (a \in \mathscr{B},\ \xi \in X \text{ such that } \delta_X(\xi) \in \mathbb{C}[G] \otimes_{\mathrm{alg}} X),$$

that is, we interpret (4.2) as a formula for the left action on X. When B is braided-commutative, (4.2) guarantees that X is a $\mathscr{B}$-bimodule, and $\mathscr{D}_B$ becomes a C*-tensor category with tensor product given by $X \otimes Y = Y \otimes_B X$. Moreover, then $U \mapsto B \times U$ becomes a unitary tensor functor $\operatorname{Rep} G \to \mathscr{D}_B$.

In the opposite direction, suppose $\mathscr{E}\colon \operatorname{Rep} G \to \mathscr{C}$ is a unitary tensor functor. Similarly to the previous subsection, we may assume that it is simply an embedding and construct a G-algebra $\mathscr{B} = \oplus_s(\bar{H}_s \otimes \mathscr{C}(\mathbb{1}, U_s))$. We then have to define a $\mathbb{C}[G]$-

module structure $\rhd\colon \mathbb{C}[G]\otimes\mathscr{B}\to\mathscr{B}$. It is defined using the map

$$\tilde{\rhd}\colon (\bar{H}_s\otimes H_s)\otimes(\bar{H}_t\otimes\mathscr{C}(\mathbb{1},U_t))\to\overline{(H_s\otimes H_t\otimes\bar{H}_s)}\otimes\mathscr{C}(\mathbb{1},U_s \mathbin{⦶} U_t \mathbin{⦶} \bar{U}_s),$$

$$(\bar{\xi}\otimes\zeta)\,\tilde{\rhd}\,(\bar{\eta}\otimes T)=\overline{(\xi\otimes\eta\otimes\overline{\rho^{-1/2}\zeta})}\otimes(\iota\otimes T\otimes\iota)\bar{R}_s,$$

by decomposing $U_s \mathbin{⦶} U_t \mathbin{⦶} \bar{U}_s$ into irreducibles.

Let us now consider the case when the target category $\mathscr{C}$ is Hilb_f. Then it can be shown that Theorem 4.6 establishes a bijection between the isomorphism classes of unitary fiber functors $\operatorname{Rep} G\to\mathrm{Hilb}_f$ and the isomorphism classes of unital Yetter–Drinfeld G-C*-algebras B such that

- the G-algebra $\mathscr{B}$ is a Hopf–Galois extension of $\mathbb{C}$ over $\mathbb{C}[G]$, meaning that $\mathscr{B}^G=\mathbb{C}1$ and the Galois map

$$\Gamma\colon\mathscr{B}\otimes\mathscr{B}\to\mathbb{C}[G]\otimes\mathscr{B},\quad x\otimes y\mapsto x_{(1)}\otimes x_{(2)}y, \tag{4.3}$$

  is bijective,
- the $\mathbb{C}[G]$-module structure $\rhd\colon\mathbb{C}[G]\otimes\mathscr{B}\to\mathscr{B}$ is completely determined by the action of G and coincides with the *Miyashita–Ulbrich action*, defined by

$$x\rhd a=\Gamma^{-1}(x\otimes 1)_1 a\Gamma^{-1}(x\otimes 1)_2.$$

In the purely algebraic setting the correspondence between fiber functors and Hopf–Galois extensions of $\mathbb{C}$ over $\mathbb{C}[G]$ was established in [52]. In the operator algebraic setting this was done in [3]. Note that in the last paper instead of bijectivity of the Galois map an equivalent condition of *full quantum multiplicity* is considered.

4.3 Dual Category

Two most natural braided-commutative Yetter–Drinfeld algebras associated with any compact quantum group G are the algebras of functions on G and on its discrete dual $\hat{G}$. Let us consider the latter in more detail.

In the Hopf–von Neumann algebraic framework, the discrete dual is defined by

$$\ell^\infty(\hat{G})=\ell^\infty\text{-}\bigoplus_{s\in\operatorname{Irr}G}B(H_s)\subset\mathscr{U}(G).$$

We have a left adjoint action α of G on $\ell^\infty(\hat{G})$ defined by

$$B(H_s)\ni T\mapsto(U_s)^*_{21}(1\otimes T)(U_s)_{21}. \tag{4.4}$$

This action is continuous only in the von Neumann algebraic sense, so in order to stay within the class of G-C*-algebras, instead of $\ell^\infty(\hat{G})$ we consider the norm closure $B(\hat{G})$ of the regular subalgebra $\ell^\infty_{\mathrm{alg}}(\hat{G}) \subset \ell^\infty(\hat{G})$. The right action $\hat{\Delta}$ of $\hat{G}$ on $\ell^\infty(\hat{G})$ makes this algebra into a unital braided-commutative Yetter–Drinfeld C*-algebra. In other words, the left $\mathbb{C}[G]$-module structure on $\ell^\infty_{\mathrm{alg}}(\hat{G})$ is defined by

$$x \rhd a = (\iota \otimes x)\hat{\Delta}(a).$$

We want to describe the corresponding C*-tensor category $\hat{\mathscr{C}} = \mathscr{C}_{B(\hat{G})}$ and the unitary tensor functor $\mathscr{F} = \mathscr{F}_{B(\hat{G})}\colon \operatorname{Rep} G \to \hat{\mathscr{C}}$. The category $\hat{\mathscr{C}}$ is the idempotent completion of the category with the same objects as in $\operatorname{Rep} G$, but with morphism sets $\hat{\mathscr{C}}(U,V) \subset B(H_U,H_V) \otimes \ell^\infty_{\mathrm{alg}}(\hat{G})$. In this picture $\mathscr{F}$ is the obvious embedding functor. For the reasons that will become apparent in a moment, it is more convenient to consider $\hat{\mathscr{C}}(U,V)$ as a subset of $\ell^\infty_{\mathrm{alg}}(\hat{G}) \otimes B(H_U,H_V)$. Then $\hat{\mathscr{C}}(U,V)$ is the set of elements $T \in \ell^\infty_{\mathrm{alg}}(\hat{G}) \otimes B(H_U,H_V)$ such that

$$V^*_{31}(\alpha \otimes \iota)(T)U_{31} = 1 \otimes T.$$

From the definition of the adjoint action α we see that an element $T \in \ell^\infty_{\mathrm{alg}}(\hat{G}) \otimes B(H_U,H_V)$ lies in $\hat{\mathscr{C}}(U,V)$ if and only if it defines a G-equivariant map $H_s \otimes H_U \to H_s \otimes H_V$ for all s. It follows that $\hat{\mathscr{C}}(U,V)$ can be identified with the space of bounded natural transformations between the functors $\iota \mathbin{\bar\otimes} U$ and $\iota \mathbin{\bar\otimes} V$ on $\operatorname{Rep} G$,

$$\mathrm{Nat}_{\mathrm{b}}(\iota \mathbin{\bar\otimes} U, \iota \mathbin{\bar\otimes} V) \cong \ell^\infty\text{-}\bigoplus_{s\in \operatorname{Irr} G} \mathrm{Hom}_G(H_s \otimes H_U, H_s \otimes H_W).$$

The composition of morphisms is defined in the obvious way as the composition of natural transformations. The tensor product of morphisms is determined by the tensor products $\iota_W \otimes \xi$ and $\xi \otimes \iota_W$ for $\xi = (\xi_X)_X \in \mathrm{Nat}_{\mathrm{b}}(\iota \mathbin{\bar\otimes} U, \iota \mathbin{\bar\otimes} V)$. These are defined by

$$(\xi \otimes \iota_W)_X = \xi_X \otimes \iota_W, \quad (\iota_W \otimes \xi)_X = \xi_{X \mathbin{\bar\otimes} W}. \tag{4.5}$$

5 Poisson Boundary

An essential ingredient in the classification of fiber functors is the notion of non-commutative Poisson boundary and its categorical counterpart, and the universality of the latter with respect to amenable tensor functors.

5.1 Noncommutative Poisson Boundary

For a finite dimensional unitary representation U of G, consider the state ϕ_U on $B(H_U)$ defined by

$$\phi_U(T) = \frac{\mathrm{Tr}(T\pi_U(\rho)^{-1})}{\dim_q U} \quad \text{for } T \in B(H).$$

If U is irreducible, it can be characterized as the unique state satisfying

$$(\iota \otimes \phi_U)(U_{21}^*(1 \otimes T)U_{21}) = \phi_U(T).$$

For our fixed representatives of irreducible representations $\{U_s\}_s$ of G, we write ϕ_s instead of ϕ_{U_s}.

When ϕ is a normal state on $\ell^\infty(\hat{G})$, we define a completely positive map P_ϕ on $\ell^\infty(\hat{G})$ by

$$P_\phi(a) = (\phi \otimes \iota)\hat{\Delta}(a).$$

If μ is a probability measure on $\mathrm{Irr}\, G$, we then define a normal unital completely positive map P_μ on $\ell^\infty(\hat{G})$ by $P_\mu = \sum_s \mu(s)P_{\phi_s}$. The space

$$H^\infty(\hat{G};\mu) = \{x \in \ell^\infty(\hat{G}) \mid x = P_\mu(x)\}$$

of P_μ-harmonic elements is called the *noncommutative Poisson boundary* [16] of $\hat{G}$ with respect to μ. This is an operator subspace of $\ell^\infty(\hat{G})$ closed under the left adjoint action α of G defined by (4.4) and the right action $\hat{\Delta}$ of $\hat{G}$ on itself by translations. It has a new product structure

$$x \cdot y = \lim_{n\to\infty} P_\mu^n(xy), \tag{5.1}$$

where the limit is taken in the strong* operator topology. With this product $H^\infty(\hat{G};\mu)$ becomes a von Neumann algebra (with the original operator space structure) [7], which can be regarded as a generalization of the classical Poisson boundary [19]. Moreover, the actions of G and $\hat{G}$ on $\ell^\infty(\hat{G})$ define continuous actions on $H^\infty(\hat{G};\mu)$ in the von Neumann algebraic sense.

5.2 Categorical Poisson Boundary

Using the categorical description of the discrete duals of compact quantum groups given in Sect. 4.3, we have a straightforward translation of noncommutative Poisson boundaries to its categorical counterpart [34].

Let $\mathscr{C}$ be a strict rigid C*-tensor category with simple unit. Consider the category $\hat{\mathscr{C}}$ defined as in Sect. 4.3 for $\mathscr{C} = \operatorname{Rep} G$. We have "partial trace" maps

$$\operatorname{tr}_U \otimes \iota \colon \mathscr{C}(U \otimes X, U \otimes Y) \to \mathscr{C}(X, Y), \quad T \mapsto \frac{1}{d(U)}(R_U^* \otimes \iota)(\iota \otimes T)(R_U \otimes \iota).$$

Using them we can define an operator P_U on $\hat{\mathscr{C}}(V, W)$ by

$$(P_U(\eta))_X = (\operatorname{tr}_U \otimes \iota)(\eta_{U \otimes X}) \in \mathscr{C}(X \otimes V, X \otimes W).$$

As before, fix representatives U_s of isomorphism classes of simple objects and write P_s for P_{U_s}. Let μ be a probability measure on $\operatorname{Irr}\mathscr{C}$. Define an operator P_μ acting on $\operatorname{Nat_b}(\iota \otimes V, \iota \otimes W)$ by $P_\mu = \sum_s \mu(s) P_s$. It is called the *Markov operator associated with* μ. A bounded natural transformation $\eta \colon \iota \otimes V \to \iota \otimes W$ is called P_μ-*harmonic* if $P_\mu(\eta) = \eta$. We denote the set of P_μ-harmonic natural transformations $\iota \otimes V \to \iota \otimes W$ by $\mathscr{P}(\mathscr{C}; \mu)(V, W)$, or just by $\mathscr{P}(V, W)$.

As is the case for usual harmonic functions, the naive product of harmonic transformation is not guaranteed to be harmonic. However, the formula (5.1) still makes sense, and $\mathscr{P}(\mathscr{C}; \mu)(V, W)$ can be considered as a morphism set of a new C*-category. We denote the subobject completion of this category by $\mathscr{P}(\mathscr{C}; \mu)$, or simply by $\mathscr{P}$.

The category $\mathscr{P}$ has the natural structure of a C*-tensor category defined similarly to $\hat{\mathscr{C}}$. Namely, at the level of objects (before the subobject completion), the tensor product $\otimes$ is the same as in $\mathscr{C}$, while the tensor product of morphisms is given by

$$\xi \otimes \eta = (\xi \otimes \iota) \cdot (\iota \otimes \eta),$$

where $\xi \otimes \iota$ and $\iota \otimes \eta$ are defined in the same way (4.5) as in $\hat{\mathscr{C}}$.

Any morphism $T \colon V \to W$ in $\mathscr{C}$ defines a bounded natural transformation $(\iota_X \otimes T)_X$ from $\iota \otimes V$ into $\iota \otimes W$, which is obviously P_μ-harmonic for every μ. This embedding of $\mathscr{C}$-morphisms into $\mathscr{P}$-morphisms defines a strict unitary tensor functor $\Pi \colon \mathscr{C} \to \mathscr{P}$.

Definition 5.1 ([34]) The pair $(\mathscr{P}, \Pi)$ is called the *Poisson boundary* of $(\mathscr{C}, \mu)$.

Note that $\hat{\mathscr{C}}(\mathbb{1}) = \ell^\infty(\operatorname{Irr}\mathscr{C})$. A probability measure μ on $\operatorname{Irr}\mathscr{C}$ is called *ergodic*, if the only P_μ-harmonic bounded functions on $\operatorname{Irr}\mathscr{C}$ are the constant functions. This means that $\mathscr{P}(\mathscr{C}; \mu)$ still has a simple unit. It can be shown that such a μ exists if and only if the intrinsic dimension function on $\mathscr{C}$ is *weakly amenable* in the terminology of [15], meaning that there exists a state on $\ell^\infty(\operatorname{Irr}\mathscr{C})$ which is P_s-invariant for all s. In this case we say that $\mathscr{C}$ is *weakly amenable*. For weakly amenable categories, the Poisson boundary has the following universal property.

Theorem 5.2 ([34]) *Let $\mathscr{C}$ be a rigid C^*-tensor category with simple unit, and μ be an ergodic probability measure on* $\operatorname{Irr}\mathscr{C}$. *Then the equality*

$$d^{\mathscr{P}(\mathscr{C};\mu)}(\Pi(X)) = \|\Gamma_X\|$$

holds for all $X \in \mathscr{C}$. Moreover, if $\mathscr{C}'$ is another C^*-tensor category with simple unit, and $\mathscr{F}: \mathscr{C} \to \mathscr{C}'$ is a unitary tensor functor such that $d^{\mathscr{C}'}(\mathscr{F}(X)) = \|\Gamma_X\|$ holds for all X, then there exists a unitary tensor functor $\mathscr{E}: \mathscr{P}(\mathscr{C};\mu) \to \mathscr{C}'$, unique up to a natural unitary monoidal isomorphism, such that $\mathscr{E}\Pi$ is naturally unitarily monoidally isomorphic to $\mathscr{F}$.

Note that the subcategory generated by the image of $\mathscr{F}$ is rigid, although the standard solutions for $\mathscr{F}(U)$ might not be in the image of $\mathscr{F}$.

The proof of the above theorem consists of two parts, which rely on very different techniques. One part establishes a universal property of the Poisson boundary among the functors defining the smallest dimension function on $\mathscr{C}$. It relies on a study of certain completely positive maps and their multiplicative domains, which can be thought of as analogues of the classical Poisson integral. The second part shows that the smallest dimension function must be $X \mapsto \|\Gamma_X\|$. The proof relies on subfactor theory and is inspired by the works of Pimsner and Popa [41, 45]. It is clear that the second part is not needed once we have at least one functor $\mathscr{F}: \mathscr{C} \to \mathscr{C}'$ such that $d^{\mathscr{C}'}(\mathscr{F}(X)) = \|\Gamma_X\|$, as is the case in the applications we describe below.

Let us draw some implications of Theorem 5.2 for $\mathscr{C} = \operatorname{Rep} G$ for a compact quantum group G. Recall that G is called *coamenable* if $\hat{G}$ is amenable. By a Kesten-type criterion this is equivalent to requiring $\|\Gamma_U\| = \dim U$ for all $U \in \operatorname{Rep} G$. Furthermore, coamenability of G implies weak amenability of $\operatorname{Rep} G$, so the above theorem can be applied. In our current setting the theorem says that there exists a universal unitary tensor functor $\Pi: \operatorname{Rep} G \to \mathscr{P}$ such that $d^{\mathscr{P}}(\Pi(U)) = \dim U$ for all U. By universality and Woronowicz's Tannaka–Krein duality this functor must correspond to a quantum subgroup of G. Since the classical and quantum dimension functions coincide only in the Kac case, it is not difficult to figure out what this quantum subgroup must be and obtain the following result.

Theorem 5.3 ([35]) *Let G be a coamenable compact quantum group, $K < G$ its maximal quantum subgroup of Kac type. Then the forgetful functor* $\operatorname{Rep} G \to \operatorname{Rep} K$ *is a universal unitary tensor functor defining the classical dimension function on* $\operatorname{Rep} G$.

Using the correspondence between the noncommutative and categorical boundaries, this implies that if G is a coamenable compact quantum group and μ is an ergodic probability measure on $\operatorname{Irr} G$, then the Poisson boundary $H^\infty(\hat{G};\mu)$ is G- and $\hat{G}$-equivariantly isomorphic to $L^\infty(G/K)$. This result was originally proved by Tomatsu [51] (he states the result in a more restricted form, but in fact his proof works in the generality we formulated).

What is more important for our deformation problems, Theorem 5.3 sometimes allows us to reduce the classification of dimension-preserving fiber functors to an easier task.

Corollary 5.4 ([35]) *With G and K as in Theorem 5.3, there is a bijective correspondence between the (natural unitary monoidal) isomorphism classes of dimension-preserving unitary fiber functors* $\operatorname{Rep} G \to \mathrm{Hilb}_\mathrm{f}$ *and those of* $\operatorname{Rep} K$. *Namely, the correspondence maps a functor* $\operatorname{Rep} K \to \mathrm{Hilb}_\mathrm{f}$ *into its composition with the forgetful functor* $\operatorname{Rep} G \to \operatorname{Rep} K$.

In other words, the natural map $H^2(\hat{K};\mathbb{T}) \to H^2(\hat{G};\mathbb{T})$ induced by the inclusion $\mathscr{U}(K) \hookrightarrow \mathscr{U}(G)$ is a bijection.

This corollary is of course void of any content when G is already of Kac type, e.g., when G is a genuine compact group. In the latter case it is, however, still possible to say something interesting about $H^2(\hat{G};\mathbb{T})$ by the theory developed by Wassermann [54, 55] and Landstad [22]. We will say a few words about it later.

Now, let us return to abstract C*-tensor categories and give another example of a computation of the categorical Poisson boundary.

Recall that in Sect. 3.2 we introduced twistings $(\operatorname{Rep} G)^\phi$ of $\operatorname{Rep} G$ by cocycles $\phi \in Z^3(\mathrm{Ch}(G);\mathbb{T})$. The same construction makes sense for arbitrary $\mathscr{C}$: given a cocycle $\phi \in Z^3(\mathrm{Ch}(\mathscr{C});\mathbb{T})$, we define new associativity morphisms $(U\otimes V)\otimes W \to U\otimes(V\otimes W)$ as those of $\mathscr{C}$ multiplied by the factor $\phi(g,h,k)$ if U, V and W have the degrees g, h and $k \in \mathrm{Ch}(\mathscr{C})$, respectively, and denote the C*-tensor category we thus obtain by $\mathscr{C}^\phi$.

The category $\hat{\mathscr{C}}$ is graded over $\mathrm{Ch}(\mathscr{C})$, since if X and Y are objects in $\mathscr{C}$ of different degrees, then there are no nonzero morphisms $X \to Y$ in $\hat{\mathscr{C}}$. It follows that for any probability measure μ on $\operatorname{Irr}\mathscr{C}$ the Poisson boundary $\mathscr{P}(\mathscr{C};\mu)$ is still graded over $\mathrm{Ch}(\mathscr{C})$. Therefore $\mathrm{Ch}(\mathscr{C})$ is a quotient of $\mathrm{Ch}(\mathscr{P}(\mathscr{C};\mu))$, so any cocycle $\phi \in Z^3(\mathrm{Ch}(\mathscr{C});\mathbb{T})$ can be viewed as a cocycle on $\mathrm{Ch}(\mathscr{P}(\mathscr{C};\mu))$ and we can consider the twisted category $\mathscr{P}(\mathscr{C};\mu)^\phi$. The functor $\Pi\colon \mathscr{C} \to \mathscr{P}(\mathscr{C};\mu)$ defines a tensor functor $\mathscr{C}^\phi \to \mathscr{P}(\mathscr{C};\mu)^\phi$, which we denote by Π^ϕ. A natural question is how $(\mathscr{P}(\mathscr{C};\mu)^\phi, \Pi^\phi)$ is related to the Poisson boundary of $(\mathscr{C}^\phi,\mu)$. We have the following result.

Theorem 5.5 ([4]) *Let $\mathscr{C}$ be a rigid C*-tensor category with simple unit, $\phi \in Z^3(\mathrm{Ch}(\mathscr{C});\mathbb{T})$, and μ be an ergodic probability measure on $\operatorname{Irr}\mathscr{C}$. Then $\Pi^\phi\colon \mathscr{C}^\phi \to \mathscr{P}(\mathscr{C};\mu)^\phi$ is a universal unitary tensor functor such that $d^{\mathscr{P}(\mathscr{C};\mu)^\phi}(\Pi^\phi(X)) = \|\Gamma_X\|$ for all $X \in \mathscr{C}^\phi$, hence it is isomorphic to the Poisson boundary of $(\mathscr{C}^\phi,\mu)$.*

This theorem is proved in [4] for $\mathscr{C} = \operatorname{Rep} G$, but the general case is essentially the same. See also [5] for a related result.

From this we obtain the following useful extension of Theorem 5.3.

Theorem 5.6 ([4]) *Let G be a coamenable compact quantum group with maximal closed quantum subgroup K of Kac type. Then*

- $\mathrm{Ch}(G)$ *is a quotient of* $\mathrm{Ch}(K)$*;*
- *for any* 3*-cocycle* $\phi \in Z^3(\mathrm{Ch}(G); \mathbb{T})$*, the forgetful functor* $(\mathrm{Rep}\, G)^\phi \to (\mathrm{Rep}\, K)^\phi$ *is a universal unitary tensor functor defining the classical dimension function on* $(\mathrm{Rep}\, G)^\phi$.

6 Quantum Groups of Lie Type

6.1 Fiber Functors on Twisted Representation Categories

Let G be a compact connected semisimple Lie group and $T \subset G$ be a maximal torus. We denote the weight and root lattices of $\mathfrak{g}$ by P and Q, respectively, and denote by $X^*(T) \subset P$ the weight lattice of T. We also denote by $\Psi = (X^*(T), \Pi, X_*(T), \Pi^\vee)$ a fixed based root datum of (G, T).

For any $q > 0$ we have canonical isomorphisms

$$\mathrm{Ch}(G_q) \cong \mathrm{Ch}(G) \cong \widehat{Z(G)} \cong X^*(T)/Q.$$

As described in Sect. 3.2, for every $\phi \in Z^3(\widehat{Z(G)}; \mathbb{T})$ we can therefore consider a new category $(\mathrm{Rep}\, G_q)^\phi$ with associativity morphisms defined by the action of ϕ.

Example 6.1 Let us give concrete examples of such cocycles ϕ. Assume $Z(G)$ is isomorphic to a cyclic group $\mathbb{Z}/n\mathbb{Z}$ (which happens, for example, when G is simple and $G \not\cong \mathrm{Spin}(4n)$). Then any 3-cocycle on $\mathrm{Ch}(G_q) \cong \mathbb{Z}/n\mathbb{Z}$ is cohomologous to the cocycle

$$\phi(a, b, c) = \omega^{\left(\lfloor \frac{a+b}{n} \rfloor - \lfloor \frac{a}{n} \rfloor - \lfloor \frac{b}{n} \rfloor\right)c} \quad (a, b, c \in \mathbb{Z}/n\mathbb{Z})$$

for some n-th root of unity ω.

The first natural question is whether the categories $(\mathrm{Rep}\, G_q)^\phi$ admit any fiber functors.

Suppose that $c \in \mathcal{U}(T \times T)$ is a $\mathbb{T}$-valued 2-cochain on the dual group $\hat{T} = X^*(T)$ such that its coboundary ∂c is invariant under Q in each variable. Then ∂c can be considered a 3-cocycle Φ^c on $\widehat{Z(G)}$ and we obtain the twisted category $(\mathrm{Rep}\, G_q)^{\Phi^c} = (\mathrm{Rep}\, G_q, \Phi^c)$ as above. Since Φ^c is the coboundary of c over $\hat{T}$, we have a unitary fiber functor $\mathscr{F}_c\colon (\mathrm{Rep}\, G_q, \Phi^c) \to \mathrm{Hilb}_\mathrm{f}$ which is identical to the canonical fiber functor on $\mathrm{Rep}\, G_q$, except that the tensor structure is given by

$$(\mathscr{F}_c)_2\colon H_U \otimes H_V \to H_{U \circledcirc V}, \quad \xi \otimes \eta \mapsto c^*(\xi \otimes \eta).$$

This functor defines a new compact quantum group G_q^c such that $\operatorname{Rep} G_q^c$ is unitarily monoidally equivalent to $(\operatorname{Rep} G_q, \Phi^c)$. Explicitly, similarly to the case of twisting by 2-cocycles, $\mathbb{C}[G_q^c] = \mathbb{C}[G]$ as coalgebras, while the new $*$-algebra structure is defined by duality from $(\mathscr{U}(G_q), c\hat{\Delta}_q(\cdot)c^*)$. Because c is defined on $\hat{T}$, the coproduct of any element $a \in \mathscr{U}(T)$ computed in $\mathscr{U}(G_q^c)$ is the same as $\hat{\Delta}_q(a) = \hat{\Delta}(a)$. In particular, T is still a closed subgroup of G_q^c.

Example 6.2 A concrete example of c can be given as follows [36]. Let r be the rank of G, and let $\tau = (\tau_1, \dots, \tau_r) \in Z(G)^r$. Take any function $c_\tau : X^*(T) \times X^*(T) \to \mathbb{T}$ satisfying

$$c_\tau(\lambda, \mu + Q) = f(\lambda, \mu), \quad c_\tau(\lambda + \alpha_i, \mu) = \langle \tau_i, \mu \rangle c_\tau(\lambda, \mu).$$

Then $\Phi^{c_\tau} = \partial(c_\tau)$ is Q-invariant in each variable and hence can be considered as a 3-cocycle on $\widehat{Z(G)}$. Moreover, any other choice of c_τ only differs by a function on $(X^*(T)/Q)^2$ and defines exactly the same quantum group $G_q^{c_\tau}$.

Let T_τ be the subgroup of $Z(G)$ generated by the components of τ. Then the algebra $\mathbb{C}[G_q^{c_\tau}]$ can be presented as a subalgebra of $\mathbb{C}[G_q] \rtimes \hat{T}_\tau$, which can be used to understand the irreducible representations and K-theory of $C(G_q^{c_\tau})$ [36].

The construction of $G_q^{c_\tau}$ is a particular example of twisting by *almost adjoint invariant cocentral actions* developed in [4].

As was already mentioned in Sect. 3.3, for $q \neq 1$, the torus T is a maximal quantum subgroup of G_q of Kac type. By Theorem 5.6 it follows that for any $\phi \in Z^3(\widehat{Z(G)}; \mathbb{T})$ the forgetful functor $(\operatorname{Rep} G_q)^\phi \to (\operatorname{Rep} T)^\phi$ is a universal unitary tensor functor defining the classical dimension function on $(\operatorname{Rep} G_q)^\phi$. This can be used in two ways. On the one hand, applying this to $\phi = \Phi^c$ we conclude that

$$H^2(\hat{G}_q^c; \mathbb{T}) \cong H^2(\hat{T}; \mathbb{T}). \tag{6.1}$$

In particular, the dimension-preserving fiber functors of G_q^c produce only quantum groups of the form $G_q^{c'}$ with $c'c^{-1} \in Z^2(\hat{T}; \mathbb{T})$. On the other hand, since it is easy to understand when $(\operatorname{Rep} T)^\phi$ admits a fiber functor, we conclude that $(\operatorname{Rep} G_q)^\phi$ admits a dimension-preserving unitary tensor functor if and only if ϕ lifts to a coboundary on $\hat{T} = X^*(T)$. This leads to the following result.

Theorem 6.3 ([4, 36]) *Let G be a compact connected semisimple Lie group with a maximal torus T, and H be a compact quantum group of G-type with representation category $(\operatorname{Rep} G_q)^\phi$ for some $q > 0$, $q \neq 1$, and $\phi \in Z^3(\widehat{Z(G)}; \mathbb{T})$. Then*

- *the cocycle ϕ is cohomologous to a cocycle of the form Φ^{c_τ} constructed in Example 6.2;*
- *the quantum group H is isomorphic to $G_q^{\theta c_\tau}$ for a cocycle $\theta \in Z^2(\hat{T}; \mathbb{T})$.*

In particular, although the construction in Example 6.2 may seem somewhat ad hoc, the theorem shows that the only other way of constructing dimension-

preserving unitary fiber functors on $(\operatorname{Rep} G_q)^\phi$ is by multiplying the cochains c_τ in that example by 2-cocycles on $\hat{T}$.

Remark 6.4

1. Since the classical and quantum dimension functions on $(\operatorname{Rep} G_q)^\phi$ coincide if and only if $q = 1$, instead of requiring $q \neq 1$ we could say that H is not of Kac type.
2. Another way of formulating the above theorem is by saying that for $q \neq 1$ the quantum groups $G_q^{\theta c_\tau}$ exhaust all quantum groups of G-type corresponding to the classes $[\phi\Phi_{\mathrm{KZ},q}] \in H^3_G(\hat{G}; \mathbb{T})$.
3. The cocycles ϕ cohomologous to the cocycles of the form Φ^{c_τ} can be abstractly characterized as those that vanish on $\wedge^3(X^*(T)/Q) \subset H_3(X^*(T)/Q; \mathbb{Z})$ [36]. In particular, when G is simple, then $\wedge^3(X^*(T)/Q) = 0$ and all cocycles satisfy this property.

Isomorphism (6.1) and Theorem 6.3 fail miserably for $q = 1$. First of all, the map $H^2(\hat{T}; \mathbb{T}) \to H^2(\hat{G}; \mathbb{T})$ maps the orbits under the action of the Weyl group into single points. Therefore as soon as the rank of some simple factor of G is at least 2, the map $H^2(\hat{T}; \mathbb{T}) \to H^2(\hat{G}; \mathbb{T})$ is not injective. More importantly, when the rank is large enough, the map is very far from being surjective. In order to formulate the precise result, let us introduce some terminology.

Let G be for a moment an arbitrary compact group. A cocycle $x \in Z^2(\hat{G}; \mathbb{T})$ is called *nondegenerate* if its cohomology class does not arise from a proper closed subgroup of G. Denote by $H^2(\hat{G}; \mathbb{T})^\times$ the (possibly empty) subset of $H^2(\hat{G}; \mathbb{T})$ consisting of classes represented by nondegenerate cocycles. The following result is essentially due to Wassermann [54, 55], although the formulation is rather taken from [30]. For finite groups and without the unitarity condition similar results were also obtained in [12, 13, 26].

Theorem 6.5 *For any compact group G we have a decomposition*

$$H^2(\hat{G}; \mathbb{T}) \cong \bigsqcup_{[K]} H^2(\hat{K}; \mathbb{T})^\times / N_G(K),$$

where the union is taken over the conjugacy classes of closed subgroups K of G and $N_G(K)$ *denotes the normalizer of K in G, which acts through the adjoint action on* $H^2(\hat{K}; \mathbb{T})$.

It can further be shown that a cocycle $x \in Z^2(\hat{K}; \mathbb{T})$ is nondegenerate if and only if the corresponding K-C*-algebra B, as described in Sect. 4, is simple (as a C*-algebra) [22, 54]. When K is finite, the structure of such K-C*-algebras is easy to understand: B must be a full matrix algebra $\operatorname{End}(H)$, and by bijectivity of the Galois map (4.3) its dimension has to be $|K|$, so the action of K must be given by an irreducible projective representation $K \to PU(H)$ of dimension $\dim H = |K|^{1/2}$.

Now, given a compact connected Lie group G with a maximal torus T, if G contains a non-abelian finite subgroup K admitting such a projective representation,

the corresponding class in $H^2(\hat{G};\mathbb{T})$ does not lie in the image of $H^2(\hat{T};\mathbb{T})$. Furthermore, as there can be a lot of such nonconjugate subgroups K, the above theorem suggests that we should not expect a simple parametrization of Kac quantum groups of G-type similar to Theorem 6.3, in particular, those that have representation category $\operatorname{Rep} G$.

6.2 Isomorphism Problem for Twisted q-Deformations

In this last section we consider the problem of classifying the quantum groups G_q^c up to isomorphism. Since we have already classified the dimension-preserving unitary fiber functors on $\operatorname{Rep} G_q^c$ (for $q \neq 1$), for this we have to understand unitary monoidal equivalences between these categories, and in particular, their autoequivalences.

We start with autoequivalences that preserve the isomorphism classes of objects. It is not difficult to see that the group of such autoequivalences of $(\operatorname{Rep} G_q)^\phi$ does not depend on ϕ [35]. Therefore these autoequivalences are described by the following result.

Theorem 6.6 ([30, 31]) *For any $q > 0$ and any compact connected semisimple Lie group G, we have a group isomorphism*

$$H^2(\widehat{Z(G)};\mathbb{T}) \cong H^2_{G_q}(\hat{G}_q;\mathbb{T})$$

induced by the inclusion $\mathscr{U}(Z(G)) \hookrightarrow \mathscr{U}(G_q)$.

In view of (6.1) the result is not surprising, at least for $q \neq 1$. The proof, however, relies on different ideas and is more constructive than that of (6.1). The main part of it shows that if $x \in Z^2_{G_q}(\hat{G}_q;\mathbb{T})$ is an invariant cocycle with the property that for any highest weight representations U_λ and U_η it acts trivially via the representation $U_\lambda \mathbin{\bar{\oplus}} U_\eta$ on the vectors of weights $\lambda + \eta$ and $\lambda + \eta - \alpha_i$ for all simple roots α_i, then $x = 1$.

Assume now that we have a unitary monoidal equivalence between $(\operatorname{Rep} G_q)^{\phi_1}$ and $(\operatorname{Rep} G_q)^{\phi_2}$. It defines an automorphism of the representation semiring $R^+(G)$. Such an automorphism must arise from an automorphism σ of the based root datum Ψ of G [24]. The automorphism σ can then be lifted to an automorphism of G_q, which has to map the class of ϕ_1 in $H^3_{G_q}(\hat{G}_q;\mathbb{T})$ into that of ϕ_2. At least for simple Lie groups it is not difficult to understand when this happens.

Proposition 6.7 ([21, 35]) *For any $q > 0$ and any compact connected simple Lie group G, the canonical map $H^3(\widehat{Z(G)};\mathbb{T}) \to H^3_{G_q}(\hat{G}_q;\mathbb{T})$ is injective , or equivalently, the map*

$$H^3(\widehat{Z(G)};\mathbb{T}) \to H^3_G(\hat{G};\mathbb{T}),\quad [\phi] \mapsto [\phi\Phi_{\mathrm{KZ},q}],$$

is injective. Furthermore, unless $G \cong \mathrm{Spin}(4n)$, the group $\mathrm{Aut}(\Psi)$ acts trivially on $H^3(\widehat{Z(G)};\mathbb{T})$.

The result is proved in [21] for $G = \mathrm{SU}(n)$. The proof for other simple groups is similar. The idea is that if U is an irreducible representation of G_q and $f\colon \mathbb{1} \to U^{\textcircled{T}k}$ is an isometric morphism which up to a phase factor can be described entirely in terms of the fusion rules, then the composition

$$U \xrightarrow{f\otimes\iota} (U \textcircled{T} U^{\textcircled{T}(k-1)}) \textcircled{T} U \xrightarrow{\phi} U \textcircled{T} (U^{\textcircled{T}(k-1)} \textcircled{T} U) \xrightarrow{\iota\otimes f^*} U$$

in $(\mathrm{Rep}\, G_q)^\phi$ is a scalar which, on the one hand, depends only on the class of ϕ in $H^3_{G_q}(\hat{G}_q;\mathbb{T})$ and, on the other hand, can be explicitly computed in terms of ϕ. It should be remarked that the proposition, as well as several other results in this section, is formulated in [35] for simply connected groups. But the proofs for their quotients are only easier. For example, the center of $\mathrm{Spin}(4n)$ is $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and in the above argument several different representations U are needed to recover the class of ϕ. But the center of any proper quotient of $\mathrm{Spin}(4n)$ is cyclic and it suffices to take only one U, namely, either the standard representation of $\mathrm{SO}_q(4n)$ or one of the spin representations.

Returning to classification of the quantum groups G_q^c, we see that, at least for simple G, unitary monoidal equivalences between their representation categories arise only from automorphisms of the root datum and from 2-cocycles on the dual of the center. Together with the classification of dimension-preserving fiber functors on $\mathrm{Rep}\, G_q^c$ given by (6.1), this leads to the following result.

Theorem 6.8 ([35]) *Let G be a compact connected simple Lie group with a maximal torus T, $q_1, q_2 \in (0,1)$, and c_1, c_2 be $\mathbb{T}$-valued 2-cochains on $X^*(T)$ such that $\partial c_1, \partial c_2$ descend to $X^*(T)/Q$. Then the quantum groups $G_{q_1}^{c_1}$ and $G_{q_2}^{c_2}$ are isomorphic if and only if $q_1 = q_2$ and there exist an element $\sigma \in \mathrm{Aut}(\Psi)$ and a $\mathbb{T}$-valued 2-cochain b on $X^*(T)/Q$ such that $c_1\sigma(c_2)^{-1}b^{-1}$ is a coboundary on $X^*(T)$.*

If we knew that the categories $(\mathrm{Rep}\, G_q)^\phi$ exhausted all rigid C*-tensor categories with fusion rules of G, then the above theorem together with Theorem 6.3 would give a classification of all non-Kac compact quantum groups of G-type, giving a partial solution of Problem 3.3. For $G = \mathrm{SU}(n)$ this is known to be the case [18, 21] (see also [42, 44] for related slightly weaker results). Furthermore, in this case the algebras $\mathbb{C}[\mathrm{SU}_q^c(n)]$ can be explicitly described in terms of generators and relations [35].

Remark 6.9 There are indications that a complete classification of non-Kac compact quantum groups of G-type should be possible beyond the case of $G = \mathrm{SU}(n)$. Specifically, in the framework of rigid semisimple $\mathbb{C}$-linear tensor categories, there are several recent classification results showing that such tensor categories with fusion rules of G are exhausted by $\mathrm{Rep}\, G_q$, with $q \in \mathbb{C}^\times$ not a nontrivial root of

unity. For example, the case of $G = \mathrm{PSp}(4)$ is known by [6], and the case of the smallest exceptional group $G = \mathrm{G}_2$ has been recently settled in [25]. For these cases it only remains to show the uniqueness of the compatible C*-structures when q is a positive real number.

Acknowledgements S.N. was supported by the European Research Council under the European Union's Seventh Framework Programme (FP/2007–2013)/ERC Grant Agreement no. 307663.

M.Y. was supported by JSPS KAKENHI Grant Number 25800058, and partially by the Danish National Research Foundation through the Centre for Symmetry and Deformation (DNRF92).

References

1. T. Banica, *Fusion rules for representations of compact quantum groups*, Exposition. Math. **17** (1999), no. 4, 313–337. MR 1734250 (2001g:46155)
2. A. A. Belavin and V. G. Drinfel'd, *The classical Yang-Baxter equation for simple Lie algebras*, Funktsional. Anal. i Prilozhen. **17** (1983), no. 3, 69–70. MR 714225 (85e:58059)
3. J. Bichon, A. De Rijdt, and S. Vaes, *Ergodic coactions with large multiplicity and monoidal equivalence of quantum groups*, Comm. Math. Phys. **262** (2006), no. 3, 703–728. MR 2202309 (2007a:46072)
4. J. Bichon, S. Neshveyev, and M. Yamashita, *Graded twisting of categories and quantum groups by group actions*, preprint (2015), `arXiv:1506.09194 [math.QA]`, to appear in Ann. Inst. Fourier.
5. J. Bichon, S. Neshveyev, and M. Yamashita, *Graded twisting of comodule algebras and module categories*, preprint (2016), `arXiv:1604.02078 [math.QA]`.
6. D. Bisch, V. F. R. Jones, and Z. Liu, *Singly generated planar algebras of small dimension, part III*, preprint (2014), `arXiv:1410.2876 [math.OA]`, to appear in Trans. Amer. Math. Soc.
7. M. D. Choi and E. G. Effros, *Injectivity and operator spaces*, J. Functional Analysis **24** (1977), no. 2, 156–209. MR 0430809 (55 3814)
8. K. De Commer and M. Yamashita, *Tannaka-Kreĭn duality for compact quantum homogeneous spaces. I. General theory*, Theory Appl. Categ. **28** (2013), No. 31, 1099–1138. MR 3121622
9. ______, *Tannaka–Kreĭn duality for compact quantum homogeneous spaces II. Classification of quantum homogeneous spaces for quantum SU(2)*, J. Reine Angew. Math. **708** (2015), 143–171. MR 3420332
10. V. G. Drinfel'd, *Hopf algebras and the quantum Yang-Baxter equation*, Dokl. Akad. Nauk SSSR **283** (1985), no. 5, 1060–1064, translation in Soviet Math. Dokl. **32** (1985), no. 1, 254–258. MR 802128 (87h:58080)
11. ______, *Quasi-Hopf algebras*, Algebra i Analiz **1** (1989), no. 6, 114–148, translation in Leningrad Math. J. **1** (1990), no. 6, 1419–1457. MR 1047964 (91b:17016)
12. P. Etingof and S. Gelaki, *Some properties of finite-dimensional semisimple Hopf algebras*, Math. Res. Lett. **5** (1998), no. 1-2, 191–197. MR 1617921 (99e:16050)
13. ______, *The classification of triangular semisimple and cosemisimple Hopf algebras over an algebraically closed field*, Internat. Math. Res. Notices **2000**, no. 5, 223–234. MR 1747109 (2001h:16039)
14. H. Furstenberg, *Boundary theory and stochastic processes on homogeneous spaces*, Harmonic analysis on homogeneous spaces (Proc. Sympos. Pure Math., Vol. XXVI, Williams Coll., Williamstown, Mass., 1972), Amer. Math. Soc., Providence, R.I., 1973, pp. 193–229. MR 0352328 (50 4815)
15. F. Hiai and M. Izumi, *Amenability and strong amenability for fusion algebras with applications to subfactor theory*, Internat. J. Math. **9** (1998), no. 6, 669–722. MR 1644299 (99h:46116)

16. M. Izumi, *Non-commutative Poisson boundaries and compact quantum group actions*, Adv. Math. **169** (2002), no. 1, 1–57. MR 1916370 (2003j:46105)
17. M. Jimbo, *A q-difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation*, Lett. Math. Phys. **10** (1985), no. 1, 63–69. MR 797001 (86k:17008)
18. B. P. A. Jordans, *A classification of $SU(d)$-type C^*-tensor categories*, Internat. J. Math. **25** (2014), no. 9, 1450081, 40. MR 3266525
19. V. A. Kaĭmanovich and A. M. Vershik, *Random walks on discrete groups: boundary and entropy*, Ann. Probab. **11** (1983), no. 3, 457–490. MR 704539 (85d:60024)
20. D. Kazhdan and G. Lusztig, *Tensor structures arising from affine Lie algebras. IV*, J. Amer. Math. Soc. **7** (1994), no. 2, 383–453. MR 1239507 (94g:17049)
21. D. Kazhdan and H. Wenzl, *Reconstructing monoidal categories*, I. M. Gel' fand Seminar, Adv. Soviet Math., vol. 16, Amer. Math. Soc., Providence, RI, 1993, pp. 111–136. MR 1237835 (95e:18007)
22. M. B. Landstad, *Ergodic actions of nonabelian compact groups*, in: Ideas and methods in mathematical analysis, stochastics, and applications (Oslo, 1988), 365–388, Cambridge Univ. Press, Cambridge, 1992. MR 1190512 (93j:46072)
23. S. Mac Lane, *Categories for the working mathematician*, second ed., Graduate Texts in Mathematics, vol. 5, Springer-Verlag, New York, 1998. MR 1712872 (2001j:18001)
24. J. R. McMullen, *On the dual object of a compact connected group*, Math. Z. **185** (1984), no. 4, 539–552. MR 733774 (85e:22010)
25. S. Morrison, E. Peters, and N. Snyder, *Categories generated by a trivalent vertex*, preprint (2015), `arXiv:1501.06869 [math.QA]`.
26. M. V. Movshev, *Twisting in group algebras of finite groups*, Funktsional. Anal. i Prilozhen. **27** (1993), no. 4, 17–23, 95; translation in Funct. Anal. Appl. **27** (1993), no. 4, 240–244 (1994). MR 1264314 (95a:16036)
27. M. Müger, *On the center of a compact group*, Int. Math. Res. Not. (2004), no. 51, 2751–2756. MR 2130607 (2005m:22003)
28. S. Neshveyev, *Duality theory for nonergodic actions*, Münster J. Math. **7** (2014), no. 2, 413–437.
29. S. Neshveyev and L. Tuset, *Notes on the Kazhdan-Lusztig theorem on equivalence of the Drinfeld category and the category of $U_q\mathfrak{g}$-modules*, Algebr. Represent. Theory **14** (2011), no. 5, 897–948. MR 2832264 (2012j:17022)
30. ———, *On second cohomology of duals of compact groups*, Internat. J. Math. **22** (2011), no. 9, 1231–1260. MR 2844801 (2012k:22010)
31. ———, *Autoequivalences of the tensor category of $U_q\mathfrak{g}$-modules*, Int. Math. Res. Not. IMRN (2012), no. 15, 3498–3508. MR 2959039
32. ———, *Compact quantum groups and their representation categories*, Cours Spécialisés [Specialized Courses], vol. 20, Société Mathématique de France, Paris, 2013. MR 3204665
33. S. Neshveyev and M. Yamashita, *Categorical duality for Yetter-Drinfeld algebras*, Doc. Math. **19** (2014), 1105–1139. MR 3291643
34. ———, *Poisson boundaries of monoidal categories*, preprint (2014), `arXiv:1405.6572 [math.OA]`.
35. ———, *Classification of non-Kac compact quantum groups of* SU(n) *type*, Int. Math. Res. Not. (2015), pp. 36.
36. ———, *Twisting the q-deformations of compact semisimple Lie groups*, J. Math. Soc. Japan **67** (2015), no. 2, 637–662. MR 3340190
37. R. Nest and C. Voigt, *Equivariant Poincaré duality for quantum group actions*, J. Funct. Anal. **258** (2010), no. 5, 1466–1503. MR 2566309
38. C. Ohn, *Quantum* SL(3, **C**)*'s with classical representation theory*, J. Algebra **213** (1999), no. 2, 721–756. MR 1673475 (2000c:17028)
39. ———, *Quantum* SL(3, $\mathbb{C}$)*'s: the missing case*, Hopf algebras in noncommutative geometry and physics, Lecture Notes in Pure and Appl. Math., vol. 239, Dekker, New York, 2005, pp. 245–255. MR 2106933 (2005h:20110)

40. V. Ostrik, *Module categories, weak Hopf algebras and modular invariants*, Transform. Groups **8** (2003), no. 2, 177–206. MR 1976459 (2004h:18006)
41. M. Pimsner and S. Popa, *Finite-dimensional approximation of pairs of algebras and obstructions for the index*, J. Funct. Anal. **98** (1991), no. 2, 270–291. MR 1111570 (92g:46076)
42. C. Pinzari, *The representation category of the Woronowicz quantum group* $\mathrm{S}_\mu\mathrm{U}(d)$ *as a braided tensor* C^**-category*, Internat. J. Math. **18** (2007), no. 2, 113–136. MR 2307417 (2008k:46212)
43. C. Pinzari and J. E. Roberts, *A duality theorem for ergodic actions of compact quantum groups on* C^**-algebras*, Comm. Math. Phys. **277** (2008), no. 2, 385–421. MR 2358289 (2008k:46203)
44. ———, *A rigidity result for extensions of braided tensor* C^**-categories derived from compact matrix quantum groups*, Comm. Math. Phys. **306** (2011), no. 3, 647–662. MR 2825504 (2012h:46125)
45. S. Popa, *C*lassification of amenable subfactors of type II, Acta Math. **172** (1994), no. 2, 163–255. MR 1278111 (95f:46105)
46. M. A. Rieffel, *Quantization and* C^**-algebras*, C^*-algebras: 1943–1993 (San Antonio, TX, 1993), Contemp. Math., vol. 167, Amer. Math. Soc., Providence, RI, 1994, pp. 66–97. MR 1292010 (95h:46108)
47. N. Yu. Reshetikhin, L. A. Takhtadzhyan, and L. D. Faddeev, *Quantization of Lie groups and Lie algebras*, Algebra i Analiz **1** (1989), no. 1, 178–206, translation in Leningrad Math. J. **1** (1990), no. 1, 193—225. MR 1015339 (90j:17039)
48. P. Schauenburg, *Turning monoidal categories into strict ones*, New York J. Math. **7** (2001), 257–265 (electronic). MR 1870871 (2003d:18013)
49. Ya. S. Soĭbel'man, *Irreducible representations of the algebra of functions on the quantum group* $\mathrm{SU}(n)$ *and Schubert cells*, Dokl. Akad. Nauk SSSR **307** (1989), no. 1, 41–45, translation in Soviet Math. Dokl. **40** (1990), no. 1, 34–38. MR 1017083 (91d:17019)
50. ———, *Algebra of functions on a compact quantum group and its representations*, Algebra i Analiz 2 (1990), no. 1, 190–212; translation in Leningrad Math. J. 2 (1991), no. 1, 161–178. MR 1049910 (91i:58053a)
51. R. Tomatsu, *A characterization of right coideals of quotient type and its application to classification of Poisson boundaries*, Comm. Math. Phys. **275** (2007), no. 1, 271–296. MR 2335776 (2008j:46058)
52. K.-H. Ulbrich, *Fibre functors of finite-dimensional comodules*, Manuscripta Math. **65** (1989), no. 1, 39–46. MR 1006625 (90e:16014)
53. A. Van Daele, *Discrete quantum groups*, J. Algebra **180** (1996), no. 2, 431–444. MR 1378538 (97a:16076)
54. A. Wassermann, *Ergodic actions of compact groups on operator algebras. II. Classification of full multiplicity ergodic actions*, Canad. J. Math. **40** (1988), no. 6, 1482–1527. MR 0990110 (92d:46168)
55. ———, *Ergodic actions of compact groups on operator algebras. I. General theory*, Ann. of Math. (2) **130** (1989), no. 2, 273–319. MR 1014926 (91e:46092)
56. S. L. Woronowicz, *Compact matrix pseudogroups*, Comm. Math. Phys. **111** (1987), no. 4, 613–665. MR 901157 (88m:46079)
57. ———, *Twisted* $\mathrm{SU}(2)$ *group. An example of a noncommutative differential calculus*, Publ. Res. Inst. Math. Sci. **23** (1987), no. 1, 117–181. MR 890482 (88h:46130)
58. ———, *Tannaka-Kreĭn duality for compact matrix pseudogroups. Twisted* $\mathrm{SU}(N)$ *groups*, Invent. Math. **93** (1988), no. 1, 35–76. MR 943923 (90e:22033)

A Homology Theory for Smale Spaces: A Summary

Ian F. Putnam

Abstract We consider Smale spaces, a particular class of hyperbolic topological dynamical systems, which include the basic sets for Smale's Axiom A systems. We present an algebraic invariant for such systems which is based on Krieger's dimension group for the special case of shifts of finite type. This theory provides a Lefschetz formula relating trace data with the number of periodic points of the system, answering a question posed by R. Bowen. The key ingredient is the existence of Markov partitions with special properties.

1 Introduction

Smale introduced the notion of an Axiom A diffeomorphism of a compact manifold [16]. For such a system, a basic set is a closed invariant subset of the non-wandering set which is irreducible in a certain sense. One of Smale's key observations was that such a set need not be a submanifold. Typically, it is some type of fractal object.

The Artin-Masur zeta function encodes the data of the periodic points for a dynamical system. Manning proved that, for any basic set of an Axiom A system, the associated Artin-Masur zeta function is rational. This led Bowen to conjecture the existence of a homology theory for such systems which had a Lefschetz-type theorem. Such a theory would immediately imply Manning's rationality result. Such a theory is given in [14]. The aim of this article is to give a short summary of this theory without proofs.

The theory takes, as its starting point, the notion of the dimension group of a shift of finite type introduced by Krieger [9] and the fundamental result of Bowen [2] that every basic set is a factor of a shift of finite type.

In an effort to give a purely topological (i.e. without reference to any smooth structure) description of the dynamics on a basic set, Ruelle introduced the notion of a Smale space [15]: a Smale space is a compact metric space, (X, d), and a homeomorphism, φ, of X, which possesses canonical coordinates of contracting

I.F. Putnam (✉)
Department of Mathematics and Statistics, University of Victoria, P.O. Box 3060, STN CSC, Victoria, BC V8W 3R4, Canada
e-mail: ifputnam@uvic.ca

T.M. Carlsen et al. (eds.), *Operator Algebras and Applications*, Abel Symposia 12,
DOI 10.1007/978-3-319-39286-8_12

and expanding directions. The precise definition involves the existence of a map $[,]$ giving canonical coordinates. Here, we review only the features necessary for the statements of our results.

There is a constant $\epsilon_X > 0$ and, for each x in X and $0 < \epsilon \leq \epsilon_X$, there are sets $X^s(x, \epsilon)$ and $X^u(x, \epsilon)$, called the local stable and unstable sets, respectively, whose product is homeomorphic to a neighbourhood of x. As ϵ varies, these form a neighbourhood base at x. Moreover, there is a constant $0 < \lambda < 1$ such that

$$d(\varphi(y), \varphi(z)) \leq \lambda d(y, z),\ y, z \in X^s(x, \epsilon_X)$$
$$d(\varphi^{-1}(y), \varphi^{-1}(z)) \leq \lambda d(y, z),\ y, z \in X^u(x, \epsilon_X)$$

The bracket $[x, y]$ is the unique point in the intersection of $X^s(x, \epsilon_X)$ and $X^u(y, \epsilon_X)$. We say that (X, φ) is non-wandering if every point of X is non-wandering for φ [8].

Stable and unstable equivalence relations are defined by

$$R^s = \{(x, y) \mid \lim_{n \to +\infty} d(\varphi^n(x), \varphi^n(y)) = 0\}$$
$$R^u = \{(x, y) \mid \lim_{n \to +\infty} d(\varphi^{-n}(x), \varphi^{-n}(y)) = 0\}.$$

We let $X^s(x)$ and $X^u(x)$ denote the stable and unstable equivalence classes of x in X. These subsets are typically not locally compact in the relative topology. However, there are natural, much nicer topologies, which are given by using the families $X^s(x', \epsilon), x' \in X^s(x), 0 < \epsilon < \epsilon_X$ and $X^u(x', \epsilon), x' \in X^u(x), 0 < \epsilon < \epsilon_X$, respectively, as bases.

The main examples of such systems are shifts of finite type (of which we will say more in a moment), hyperbolic toral automorphisms, solenoids, substitution tiling spaces (under some hypotheses) and, most importantly, the basic sets for Smale's Axiom A systems [3, 16].

Let (Y, ψ) and (X, φ) be Smale spaces. A *factor map* from (Y, ψ) to (X, φ) is a function $\pi : Y \to X$ which is continuous, surjective and satisfies $\pi \circ \psi = \varphi \circ \pi$. It is clear that, for any y in Y, $\pi(Y^s(y)) \subset X^s(\pi(y))$ and $\pi(Y^u(y)) \subset X^u(\pi(y))$. David Fried [7] defined π to be s-resolving (or u-resolving) if, for every y in Y, the restriction of π to $Y^s(y)$ (or to $Y^u(y)$, respectively) is injective. We say that π is *s-bijective* (or *u-bijective*) if, for every y in Y, π is a bijection from $Y^s(y)$ to $X^s(\pi(y))$ (or from $Y^u(y)$ to $X^u(\pi(y))$, respectively). This actually implies that π is a local homeomorphism from the local stable sets (or unstable sets, respectively) in Y to those in X. In the case that (X, φ) is non-wandering, s-resolving (u-resolving) and s-bijective (u-bijective, respectively) are equivalent.

2 Shifts of Finite Type

Shifts of finite type are described in detail in [10]. We consider a finite directed graph G. This consists of a finite vertex set G^0, a finite edge set G^1 and maps i, t (for initial and terminal) from G^1 to G^0. The associated shift space

$$\Sigma_G = \{e = (e^k)_{k \in \mathbb{Z}} \mid e^k \in G^1, t(e^k) = i(e^{k+1}), k \in \mathbb{Z}\} \tag{2.1}$$

consists of all bi-infinite paths in G, specified as an edge list. The map σ is the left shift on Σ_G defined by $\sigma(e)^k = e^{k+1}$, for all e in Σ_G and k in $\mathbb{Z}$. By a shift of finite type, we mean any system topologically conjugate to (Σ_G, σ), for some graph G. This is not the usual definition, but is equivalent to it (see Theorem 2.3.2 of [10]). For e in Σ_G, we define $e^{[K,L]} = (e^K, \ldots .e^L)$, for all integers $K \leq L$ and also $e^{[K+1,K]} = t(e^K)$, for convenience. We then use the metric

$$d(e,f) = \inf\{1, 2^{-N-1} \mid e^{[1-N,N]} = f^{[1-N,N]}, N \geq 0\},$$

for e,f in Σ_G. We observe that such systems are Smale spaces by noting that the bracket operation is defined, with $\epsilon_X = 1/2$, as follows. For e,f in Σ_G, $[e,f]$ is defined if $t(e^0) = t(f^0)$ and then it is the sequence, $(\ldots, f^{-1}, f^0, e^1, e^2, \ldots)$. For any $l \geq 1$ and e_0 in Σ_G, the local stable and unstable sets are given by

$$\Sigma_G^s(e_0, 2^{-l}) = \{e \in \Sigma \mid e^k = e_0^k, k \geq -l\},$$
$$\Sigma_G^u(e_0, 2^{-l}) = \{e \in \Sigma \mid e^k = e_0^k, k \leq l\}.$$

It is a simple matter to see the constant $\lambda = 1/2$ will satisfy the axioms.

Theorem 2.1 *Shifts of finite type are exactly the zero-dimensional (i.e. totally disconnected) Smale spaces.*

The fundamental rôle of shifts of finite type is demonstrated by the following universal property, due to Bowen [2]. This builds on earlier results by many others, including Adler and Weiss and Sinai.

Theorem 2.2 (Bowen) *Let (X, φ) be a non-wandering Smale space. Then there exists a non-wandering shift of finite type (Σ, σ) and a finite-to-one factor map*

$$\pi : (\Sigma, \sigma) \to (X, \varphi).$$

3 Krieger's Dimension Group Invariant

Krieger [9] defined the past and future dimension groups of a shift of finite, (Σ, σ), as follows. Consider $\mathscr{D}^s(\Sigma, \sigma)$ to be the collection of compact, open subsets of $\Sigma^s(e, \epsilon)$, as e varies over Σ and $0 < \epsilon \leq \epsilon_\Sigma$. We let $\sim$ denote the smallest equivalence relation on $\mathscr{D}^s(\Sigma, \sigma)$ such that

1. If E, F are in $\mathscr{D}^s(\Sigma, \sigma)$ with $[E, F] = F$ and $[F, E] = E$ (meaning both are defined), then $E \sim F$,
2. If $E, F, \varphi(E)$ and $\varphi(F)$ are all in $\mathscr{D}^s(\Sigma, \sigma)$, then $E \sim F$ if and only if $\varphi(E) \sim \varphi(F)$.

As an example, it is a simple matter to check that for any e_0, f_0 and integer $l \geq 1$, the sets $\Sigma^s_G(e_0, 2^{-l})$ and $\Sigma^s_G(f_0, 2^{-l})$ (as described just prior to Theorem 2.2) are equivalent if $i(e_0^{-l}) = i(f_0^{-l})$.

We generate a free abelian group on the equivalence classes of elements, E, of $\mathscr{D}^s(\Sigma, \sigma)$ (denoted $[E]$) subject to the additional relation that $[E \cup F] = [E] + [F]$, if $E \cup F$ is in $\mathscr{D}^s(\Sigma, \sigma)$ and E and F are disjoint. The result is denoted by $D^s(\Sigma, \sigma)$. There is an analogous definition of $D^u(\Sigma, \sigma)$.

Theorem 3.1 (Krieger [9]) *Let G be a finite directed graph and (Σ_G, σ) be the associated shift of finite type. Then $D^s(\Sigma_G, \sigma)$ (or $D^u(\Sigma_G, \sigma)$, respectively) is isomorphic to the inductive limit of the sequence*

$$\mathbb{Z}G^0 \xrightarrow{\gamma^s} \mathbb{Z}G^0 \xrightarrow{\gamma^s} \cdots$$

where $\mathbb{Z}G^0$ denotes the free abelian group on the vertex set G^0 and the map $\gamma^s(v) = \sum_{t(e)=v} i(e)$, for any v in G^0 (or respectively, replacing γ^s by γ^u, whose definition is the same, interchanging the rôles of i and t).

The first crucial result for the development of our theory is the following functorial property of D^s and D^u, which can be found in [4].

Theorem 3.2 *Let (Σ, σ) and (Σ', σ) be shifts of finite type and let $\pi : (\Sigma, \sigma) \to (\Sigma', \sigma)$ be a factor map.*

1. If π is s-bijective, then there are natural homomorphisms

$$\begin{aligned} \pi^s &: D^s(\Sigma, \sigma) \to D^s(\Sigma', \sigma), \\ \pi^{u*} &: D^u(\Sigma', \sigma) \to D^u(\Sigma, \sigma). \end{aligned}$$

2. If π is u-bijective, then there are natural homomorphisms

$$\begin{aligned} \pi^u &: D^u(\Sigma, \sigma) \to D^u(\Sigma', \sigma), \\ \pi^{s*} &: D^s(\Sigma', \sigma) \to D^s(\Sigma, \sigma). \end{aligned}$$

The idea is simple enough: in the covariant case, the induced map sends the class of a set E in $\mathscr{D}^s(\Sigma, \sigma)$ to the class of $\pi(E)$, while in the contravariant case, the map sends the class of E' in $\mathscr{D}^s(\Sigma', \sigma)$ to the class of $\pi^{-1}(E')$. The latter is not correct since $\pi^{-1}(E')$ may not even be contained in a single stable equivalence class, but it suffices that it may be written as a finite union of elements of $\mathscr{D}^s(\Sigma, \sigma)$. These ideas can be made precise under the stated hypotheses.

4 s/u-Bijective Pairs

The key ingredient in our construction is the following notion.

Definition 4.1 Let (X, φ) be a Smale space. An *s/u-bijective pair*, π, for (X, φ) consists of Smale spaces (Y, ψ) and (Z, ζ) and factor maps

$$\pi_s : (Y, \psi) \to (X, \varphi), \qquad \pi_u : (Z, \zeta) \to (X, \varphi)$$

such that

1. $Y^u(y, \epsilon)$ is totally disconnected, for all y in Y and $0 < \epsilon \leq \epsilon_Y$,
2. π_s is s-bijective,
3. $Z^s(z, \epsilon)$ is totally disconnected, for all z in Z and $0 < \epsilon \leq \epsilon_Z$,
4. π_u is u-bijective.

To summarize the idea in an informal way, the space Y is an extension of X, where the local unstable sets are totally disconnected, while the local stable sets are homeomorphic to those in X. The existence of such s/u-bijective pairs, at least for non-wandering (X, φ), can be deduced from the results of [13] or [6]. It can be viewed as a coordinate-wise version of Bowen's theorem.

If $(Y, \psi, \pi_u, Z, \zeta, \pi_s)$ is an s/u-bijective pair for (X, φ), then we form the fibred product $\Sigma(\pi) = \{(y, z) \in Y \times Z \mid \pi_s(y) = \pi_u(z)\}$ with the map $\sigma = \psi \times \zeta|_{\Sigma(\pi)}$ and the two canonical projections $\rho_u : \Sigma(\pi) \to Y$ and $\rho_s : \Sigma(\pi) \to Z$. It is an easy matter to check that the former is u-bijective while the latter is s-bijective. It follows that both local stable and local unstable sets in $\Sigma(\pi)$ are totally disconnected and, hence, that $(\Sigma(\pi), \sigma)$ is a shift of finite type. Thus, we recover Bowen's result 2.2 with the additional condition that π may be factored as $\pi = \pi_s \circ \rho_u = \pi_u \circ \rho_s$, with maps with the additional properties described. We remark that it is not true that an arbitrary map π as in Theorem 2.2 has such a decomposition.

Theorem 4.2 *If (X, φ) is non-wandering, then there exists an s/u-bijective pair for (X, φ).*

Definition 4.3 Let $\pi = (Y, \psi, \pi_u, Z, \zeta, \pi_s)$ be an s/u-bijective pair for (X, φ). For each $L, M \geq 0$, we define

$$\Sigma_{L,M}(\pi) = \{(y_0, \ldots, y_L, z_0, \ldots, z_M) \mid y_l \in Y, z_m \in Z,$$
$$\pi_s(y_l) = \pi_u(z_m), 0 \leq l \leq L, 0 \leq m \leq M\}$$

For simplicity, we also denote $\Sigma_{0,0}(\pi)$ by $\Sigma(\pi)$.

We define $\sigma_{L,M} : \Sigma_{L,M}(\pi) \to \Sigma_{L,M}(\pi)$ by

$$\sigma_{L,M}(y_0, \ldots, y_L, z_0, \ldots, z_M) = (\psi(y_0), \ldots, \psi(y_L), \zeta(z_0), \ldots, \zeta(z_M)).$$

For $L \geq 1$ and $0 \leq l \leq L$, we let $\delta_{l,} : \Sigma_{L,M}(\pi) \to \Sigma_{L-1,M}(\pi)$ be the map which deletes entry y_l. Similarly, the map $\delta_{,m} : \Sigma_{L,M}(\pi) \to \Sigma_{L,M-1}(\pi)$ deletes entry z_m, for $M \geq 1, 0 \leq m \leq M$.

The important properties of these systems and maps is summarized as follows.

Theorem 4.4

1. *For every $L, M \geq 0$, $(\Sigma_{L,M}(\pi), \sigma_{L,M})$ is a shift of finite type.*
2. *For $L \geq 1$, $0 \leq l \leq L$, the map $\delta_{l,}$ is an s-bijective factor map.*
3. *For $M \geq 1$, $0 \leq m \leq M$, the map $\delta_{,m}$ is a u-bijective factor map.*

5 Homology

There are actually two homology theories here. One, based on the dimension group D^s will be denoted by H^s_* and the other, based on D^u, will be denoted by H^u_*. We will concentrate on the former for the remainder of this note.

It is worth noting that, if (X, φ) is a Smale space, then so is (X, φ^{-1}), although 'stable' in the former is the same as 'unstable' in the latter. Following this idea carefully through the theory, one can check that $H^u(X, \varphi)$ is naturally isomorphic to $H^s(X, \varphi^{-1})$.

Let (X, φ) be a Smale space and suppose that $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ is an s/u-bijective pair for (X, φ). By applying Krieger's invariant to each of the shifts of finite type, $\Sigma_{L,M}(\pi)$, and the maps between them induced by the maps $\delta_{l,}, \delta_{,m}, 0 \leq l \leq L, 0 \leq m \leq M$, one obtains a double complex. In a standard way, we can take the homology of this double complex as follows.

We define

$$C^s(\pi)_N = \bigoplus_{L-M=N} D^s(\Sigma_{L,M}(\pi), \sigma_{L,M})$$

for every N in $\mathbb{Z}$ and a boundary map $d^s(\pi)_N : C^s_N(\pi) \to C^s_{N-1}(\pi)$ by

$$d^s(\pi)_N | D^s(\Sigma_{L,M}, \sigma_{L,M}) = \sum_{l=0}^{L} (-1)^l \delta^s_{l,} + \sum_{m=0}^{M+1} (-1)^{m+L} \delta^{s*}_{,m}.$$

where, in the special case $L = 0$, we set $\delta^s_{0,} = 0$.

Unfortunately, this complex is rather large and we will replace it with others that are more manageable. In more technical terms, the fact that Krieger's invariant is covariant for the maps $\delta_{l,}$ but contravariant for the maps $\delta_{,m}$ mean that the double complex is a 'second quadrant' double complex, despite our choice of indexing $L, M \geq 0$. In simplicial homology, there are two complexes which may be used. One is called the ordered complex (which is analogous to one which we have above) and the second is called the alternating complex. In the former, the N-chains are constructed from $N+1$-tuples of vertices, all in a single simplex. In the latter, the N-chains are constructed from the N-simplices. The difference is that the former allows for repetition of vertices (i.e. degenerate simplices) and distinguishes between permutations of the vertices. The latter does not consider the order, except for orientation.

In our case, we have actions of the group permutation group $S_{L+1} \times S_{M+1}$ on $(\Sigma_{L,M}(\pi), \sigma_{L,M})$ and hence on the invariant $D^s(\Sigma_{L,M}(\pi), \sigma_{L,M})$. We write these actions on the right. We form $D^s_{\mathscr{Q},}(\Sigma_{L,M}(\pi), \sigma_{L,M})$ which the quotient of $D^s(\Sigma_{L,M}(\pi), \sigma_{L,M})$ by the subgroup of all elements which are fixed under some transposition and all elements of the form $a - sgn(\alpha)a \cdot (\alpha, 1)$, where α is in S_{L+1}. We also form a subgroup $D^s_{,\mathscr{A}}(\Sigma_{L,M}(\pi), \sigma_{L,M})$ of $D^s(\Sigma_{L,M}(\pi), \sigma_{L,M})$ consisting of those elements a such that $a \cdot (1, \beta) = sgn(\beta)a$, for all β in S_{M+1}. Finally, we form a subgroup of the former, also a quotient of the latter, $D^s_{\mathscr{Q},\mathscr{A}}(\Sigma_{L,M}(\pi), \sigma_{L,M})$, which incorporates the action of $S_{L+1} \times S_{M+1}$. We define complexes $(C^s_{\mathscr{Q},}(\pi), d^s_{\mathscr{Q},}(\pi))$, $(C^s_{,\mathscr{A}}(\pi), d^s_{,\mathscr{A}}(\pi))$ and $(C^s_{\mathscr{Q},\mathscr{A}}(\pi), d^s_{\mathscr{Q},\mathscr{A}}(\pi))$ using these groups instead.

The most obvious advantage of these new complexes is the following. Each s-bijective or u-bijective factor map is finite-to-one. Indeed, there are constants L_0 and M_0 such that

$$\#\pi_s^{-1}\{x\} \leq L_0, \#\pi_u^{-1}\{x\} \leq M_0$$

for all x in X. It follows that $D^s_{\mathscr{Q},}(\Sigma_{L,M}(\pi), \sigma_{L,M}) = 0$ if $L \geq L_0$, $D^s_{,\mathscr{A}}(\Sigma_{L,M}(\pi), \sigma_{L,M}) = 0$ if $M \geq M_0$ and $D^s_{\mathscr{Q},\mathscr{A}}(\Sigma_{L,M}(\pi), \sigma_{L,M}) = 0$ if either $L \geq L_0$ or $M \geq M_0$.

Standard techniques in homological algebra yield the following.

Theorem 5.1 *Let π be an s/u-bijective pair for the Smale space (X, φ). We have a commutative diagram of chain complexes and chain maps as shown:*

$$\begin{tikzcd}
(C^s_{,\mathscr{A}}(\pi), d^s_{,\mathscr{A}}(\pi)) \arrow[r, "J"] \arrow[d, "Q_{\mathscr{A}}"'] & (C^s(\pi), d^s(\pi)) \arrow[d, "Q"'] \\
(C^s_{\mathscr{Q},\mathscr{A}}(\pi), d^s_{\mathscr{Q},\mathscr{A}}(\pi)) \arrow[r, "J_{\mathscr{Q}}"] & (C^s_{\mathscr{Q},}(\pi), d^s_{\mathscr{Q},}(\pi))
\end{tikzcd}$$

Moreover, the maps $Q_{\mathscr{A}}$ and $J_{\mathscr{Q}}$ both induce isomorphisms on homology.

So the three new complexes all have the same homology. It seems likely that the original complex also has the same homology, but this does not seem to follow easily from standard techniques.

Definition 5.2 Let (X, φ) be a Smale space and π be an s/u-bijective pair for (X, φ). We define $H^s(\pi)$ to be the homology of the complex $(C^s_{\mathscr{Q},\mathscr{A}}(\pi), d^s_{\mathscr{Q},\mathscr{A}}(\pi))$. There is an analogous definition of $H^u(\pi)$.

We remark that the definition is valid for any Smale space which has an s/u-bijective pair, which includes all non-wandering Smale spaces.

6 Properties

We want to establish some basic properties of our theory. The first crucial result is the following. It is stated in a slightly informal manner, but one which conveys the main idea.

Theorem 6.1 *Let (X, φ) be a Smale space and π be an s/u-bijective pair for (X, φ). $H^s_*(\pi)$ is independent of the s/u-bijective pair $\pi = (Y, \psi, \pi_s, Z, \zeta, \pi_u)$ and depends only on (X, φ).*

Henceforth, we denote $H^s_*(\pi)$ by $H^s_*(X, \varphi)$ instead.

Theorem 6.2 *The homology theory H^s_* is functorial in the following sense. If $\rho : (X, \varphi) \to (X', \varphi')$ is an s-bijective factor map, then there are induced group homomorphisms*

$$\rho^s : H^s_N(X, \varphi) \to H^s_N(X', \varphi'),$$

for all N in $\mathbb{Z}$. If the map ρ is a u-bijective factor, then there are induced group homomorphisms

$$\rho^{s*} : H^s_N(X', \varphi') \to H^s_N(X, \varphi),$$

for all N in $\mathbb{Z}$.

The following is an easy consequence of our earlier observation that $D^s_{\mathcal{Q},\mathcal{A}}(\Sigma_{L,M}(\pi), \sigma_{L,M})$ is non-zero for only finitely many values of L and M and the fact that Krieger's invariant is always a finite rank abelian group.

Theorem 6.3 *For any Smale space (X, φ) which has an s/u-bijective pair, the groups $H^s_N(X, \varphi)$ are finite rank and are non-zero for only finite many values of N.*

Finally, we have the following analogue of the Lefschetz formula. Given (X, φ), we can regard φ^{-1} as a factor map from this system to itself. It is both s-bijective and u-bijective and so, by Theorem 6.2, induces an automorphism of our invariant, denoted $(\varphi^{-1})^s$. The proof of the following result, already known in the case of shifts of finite type, uses ideas of Manning [11].

Theorem 6.4 *For any non-wandering Smale space (X, φ) and $p \geq 1$, we have*

$$\sum_{N \in \mathbb{Z}} (-1)^N Tr[((\varphi^{-1})^s \otimes 1_{\mathbb{R}})^p : H^s_N(X, \varphi) \otimes \mathbb{Q} \to H^s_N(X, \varphi) \otimes \mathbb{Q}]$$
$$= \#\{x \in X \mid \varphi^p(x) = x\}.$$

7 Examples

We present four examples where the computations above may be carried out quite explicitly. All of the examples are computed using the double complex $(C^s_{\mathcal{Q},\mathcal{A}}(\pi), d^s_{\mathcal{Q},\mathcal{A}}(\pi))$.

Example 7.1 Suppose (Σ, σ) is a shift of finite type. In this case, an s/u-bijective pair is just $(Y, \psi) = (Z, \zeta) = (\Sigma, \sigma)$. Only the $0, 0$-term in the double complex is non-zero and it is just $D^s(\Sigma, \sigma)$. Hence, $H^s_N(\Sigma, \sigma)$ is just $D^s(\Sigma, \sigma)$, for $N = 0$, and zero otherwise.

Example 7.2 For $m \geq 2$, let (X, φ) be the m^∞-solenoid. More specifically, we let

$$X = \{(z_0, z_1, \ldots) \mid z_n \in \mathbb{T}, z_n = z^m_{n+1}, n \geq 0\},$$

with the map

$$\varphi(z_0, z_1, \ldots) = (z^m_0, z^m_1, z^m_2, \ldots),$$

for $(z_0, z_1, \ldots)$ in X. In this case, there is an s-bijective factor map onto (X, φ) from the full m-shift (i.e. G is the graph with one vertex and m edges). The simplest s/u-bijective pair here is $(Y, \psi) = (\Sigma_G, \sigma)$ and $(Z, \zeta) = (X, \varphi)$. The only non-zero groups in the double complex occur for (L, M) equal to $(0, 0)$ and $(1, 0)$ and these are $\mathbb{Z}[m^{-1}]$ and $\mathbb{Z}$, respectively. The boundary maps are all zero (only one needs to be computed) and $H^s_N(X, \varphi)$ is isomorphic to $\mathbb{Z}[m^{-1}]$, for $N = 0$, $\mathbb{Z}$, for $N = 1$ and zero for all other N.

This is a special case of a one-dimensional solenoid. The general case is described and analyzed in [1].

Example 7.3 Let $n > m > 1$ be relatively prime. Let X be the mn-solenoid as above and define

$$\varphi(z_0, z_1, \ldots) = (z_1^{n^2}, z_2^{n^2}, z_3^{n^2}, \ldots),$$

Note that X is the dual of the discrete group $\mathbb{Z}[\frac{1}{nm}]$ and φ is the dual of the automorphism which is multiplication by $\frac{n}{m}$. We refer to (X, φ) as an $\frac{n}{m}$-solenoid. Here, the local stable sets are totally disconnected (in fact, they are open sets in the field of n-adic numbers) while the local unstable sets are of the form $(-t, t) \times C$, where C is totally disconnected. Here, (Y, σ) is the full shift on n symbols, while $Z = X$.

We have $H_N^s(X, \varphi)$ is isomorphic to $\mathbb{Z}[1/n]$ for $N = 0$, $\mathbb{Z}[1/m]$, for $N = 1$, and 0 for all other values of N. For complete details, see [5].

Example 7.4 Let X be the 2-torus, $\mathbb{T}^2$, and φ be the hyperbolic automorphism determined by the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

The systems (Y, ψ) and (Z, ζ) are both DA (or derived from Anosov) systems. Moreover, the shift of finite type $(\Sigma_{0,0}(\pi), \sigma_{0,0})$ arises from the Markov partition with three rectangles which appears in many dynamics texts, for example [10]. The only non-zero terms in the double complex are in positions $(L, M) = (0, 0), (1, 0), (0, 1)$ and $(1, 1)$. The calculation yields $H_N^s(X, \varphi)$ is $\mathbb{Z}$ for $N = 1$ and $N = -1$ and is $\mathbb{Z}^2$ for $N = 0$. Notice that the homology coincides with that of the torus, except with a dimension shift.

Example 7.5 There is an example via inverse limits roughly based on the Sierpinski gasket. Its local stable sets are totally disconnected while its local unstable sets look like the Sierpinski gasket. Its homology is the same as that of the full 3-shift. The example is given in [17], although the homology calculations have not appeared as yet.

8 Concluding Remarks

Remark 8.1 It is certainly a natural question to ask whether this theory can be computed from other (already existing) machinery. A more specific question would be to relate our homology to, say, the Cech cohomology of the classifying space of the topological equivalence relation R^s. (For a discussion of the topology, see [12].)

There are examples, such as the first three above, where they are different, but only up to a dimension shift (depending on the space under consideration).

Remark 8.2 An important motivation in the construction of this theory was to compute the K-theory of certain C^*-algebras associated with the Smale space (X, φ). See [12] for a discussion of these C^*-algebras. At present, there seems to be a spectral sequence which relates the two; this work is still in progress.

Acknowledgements I would like to thank the referee for a number of helpful comments.

References

1. M. Amini, I.F. Putnam and S. Saeidi Gholikandi, *Homology for one-dimensional solenoids*, to appear, Math. Scand.
2. R. Bowen, *Markov partitions for Axiom A diffeomorphisms*, Amer. J. Math. **92** (1970), 725–747.
3. R. Bowen, *On Axiom A Diffeomorphisms*, AMS-CBMS Reg. Conf. **35**, Providence, 1978.
4. M. Boyle, B. Marcus and P. Trow, *Resolving maps and the dimension group for shifts of finite type*, Memoirs A.M.S. No. 377 **70** (1987).
5. N. Burke and I.F. Putnam, *Markov partitions and homology for $\frac{n}{m}$-solenoids*, Ergodic Theory and Dynamical Systems, to appear.
6. T. Fisher, *Resolving extensions of finitely presented systems*, Acta. Applic. Math. **126** (2013), 131–163.
7. D. Fried, *Finitely presented dynamical systems*, Ergod. Th. & Dynam. Sys. **7** (1987), 489–507.
8. A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge Univ. Press., Cambridge, 1995.
9. W. Krieger, *On dimension functions and topological Markov chains*, Inventiones Math. **56** (1980), 239–250.
10. D. Lind and B. Marcus, *Symbolic Dynamics and Coding*, Cambridge Univ. Press, Cambridge, 1995.
11. A. Manning, *Axiom A diffeomorphisms have rational zeta functions*, Bull. London Math. Soc. **3** (1971), 215–220.
12. I. F. Putnam, *C^*-algebras from Smale spaces*, Canad. J. Math. **48** (1996), 175–195.
13. I. F. Putnam, *Lifting factor maps to resolving maps*, Israel J. Math. **146** (2005), 253–280.
14. I. F. Putnam, *A homology theory for Smale spaces*, Mem. A.M.S. 232 (2014).
15. D. Ruelle, *Thermodynamic Formalism*, Encyclopedia of Math. and its Appl. **5**, Addison-Wesley, Reading, 1978.
16. S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747–817.
17. S. Wieler, *Smale spaces via inverse limits*, Ergod. Th. & Dynam, Sys. **34** (2014), 2066–2092.

On the Positive Eigenvalues and Eigenvectors of a Non-negative Matrix

Klaus Thomsen

Abstract The paper develops the version of the classical Perron-Frobenius theory which is appropriate for the study of KMS weights of generalized gauge actions on simple graph C^*-algebras.

1 Introduction

Recent work on KMS states and weights on graph C^*-algebras has uncovered an intimate relation to positive eigenvalues and eigenvectors for non-negative matrices naturally associated to the graph and the one-parameter action. Specifically, for the gauge action on a graph C^*-algebra there is a non-negative matrix B over the vertexes V in the graph, such that KMS weights corresponding to the inverse temperature $\beta \in \mathbb{R}$ are in bijective correspondence with the non-zero non-negative vectors ξ with the properties that

$$\sum_{w \in V} B_{vw} \xi_w \leq e^{\beta} \xi_v \tag{1.1}$$

for all vertexes v, and

$$\sum_{w \in V} B_{vw} \xi_w = e^{\beta} \xi_v \tag{1.2}$$

when v is not a sink and does not emit infinitely many edges. If a state, rather than a weight is sought for, one should in addition insist that $\sum_v \xi_v = 1$. The same kind of equations, but with different matrices determine also the gauge invariant KMS weights and states for more general actions on graph C^*-algebras. See [2, 4, 8].

Finding the solutions to (1.1) and (1.2) is in general a highly non-trivial task. The literature on positive eigenvalues and eigenfunctions of a non-negative matrix is enormous, and for finite graphs there are in fact results available that can be

K. Thomsen (✉)
Institut for Matematik, Aarhus University, Ny Munkegade, 8000 Aarhus C, Denmark
e-mail: matkt@math.au.dk

T.M. Carlsen et al. (eds.), *Operator Algebras and Applications*, Abel Symposia 12,
DOI 10.1007/978-3-319-39286-8_13

used to determine the possible values of β and for each β get a description of the corresponding vectors ξ, albeit with some additional work, cf. [3]. For infinite matrices this is no longer the case—very far from. The most fundamental questions prompted by the connection to KMS states and weights are those that I guess come to the mind of any mathematician:

(a) For which β are there non-zero non-negative solutions to the equations ?
(b) How does the structure of the solutions vary with β ?

From the theory of countable state Markov chains it is known that these questions are often extremely hard to answer already when the matrix is irreducible and stochastic, but also that the structure of the solutions can be very rich and interesting.

The concern here is that a general setting for an approach to the above problem is missing, although the theory of harmonic and super-harmonic functions of countable state Markov chains comes close. It is the purpose with the present paper to provide a framework for the work on the problem when the graph C^*-algebra is simple, or more precisely when the graph is cofinal. As far as I know, no one has developed the theory in this setting. It involves a possibly infinite matrix B as above for which the underlying directed graph need not be strongly connected, and what is sought are neither exactly the harmonic functions of $e^{-\beta}B$ nor exactly the super-harmonic functions. But it is something in between and the cofinality of the associated graph is a property which can substitute for the often assumed strong connectivity. What I show is how the known methods, which typically deal with harmonic or super-harmonic functions (or vectors) of a non-negative matrix, often stochastic or sub-stochastic for which the underlying digraph is strongly connected, can be modified to the yield the desired framework. As a consequence the paper is expository because the ideas I present are known. The purpose is to show how the tools must be arranged in order to address the problem above. When the graph is strongly connected with finite out-degree at every vertex, everything I present can be obtained from the theory of countable state Markov chains in combination with the work of Vere-Jones, [9], although the translation may not be straightforward. In the more general cofinal case, and in the presence of infinite emitters, some non-trivial adjustments to the methods developed for Markov chains must be performed, and rather than describing first the Markov chain results and then the adjustments, I have chosen to give a self-contained account, requiring no knowledge of Markov chains or random walks. I make no claim of originality for the underlying ideas, but I hope that the presentation will be useful for mathematicians interested in KMS weights and states on C^*-algebras. Needless to say, I would be happy if workers from other fields of mathematics also find it worthwhile. I have kept the list of references to an absolute minimum by quoting only my own sources. The reason for this is that I am unable to point to the original sources of the ideas presented, not for lack of good will, but out of ignorance. I therefore choose to follow the principle 'none mentioned, none forgotten'. I apologise to anyone offended by this.

Once the right setup is in place it is easy to begin to harvest results from the theory of Markov chains. As an illustration of this I use in the final section the theorem on convergence to the boundary for a countable state Markov chain to obtain the

general and abstract description of extremal solutions to Eqs. (1.1) and (1.2) in the cofinal case.

2 The Setting

Let V be a countable set and

$$V \times V \ni (v, w) \mapsto A_{vw} \in [0, \infty)$$

a non-negative matrix A over V. We can then consider the directed graph G with vertexes V such that there is an arrow from $v \in V$ to $w \in V$ if and only if $A_{vw} \neq 0$. Let E denote the set of edges (or arrows) in G. When μ is an edge, or more generally a finite path in G, we denote by $s(\mu)$ and $r(\mu)$ its initial and terminal vertex, respectively. Let V_∞ denote the union of the sinks and the infinite emitters in V, i.e.

$$V_\infty = \left\{v \in V : \ \#s^{-1}(v) \in \{0, \infty\} \right\}.$$

A subset $H \subseteq V$ is *hereditary* when $e \in E, s(e) \in H \Rightarrow r(e) \in H$, and *saturated* when

$$v \in V \backslash V_\infty, \ r(s^{-1}(v)) \subseteq H \ \Rightarrow \ v \in H.$$

We assume that A is *cofinal* in the sense that V does not contain any subsets that are both hereditary and saturated other than $\emptyset$ and V.

Lemma 2.1 *Assume that A is cofinal. Let $e_1e_2e_3\cdots$ be an infinite path in G. For every $v \in V$ there is an $i \in \mathbb{N}$ and a finite path μ in G such that $s(\mu) = v$ and $r(\mu) = s(e_i)$.*

Proof The set of vertexes v which do not have the stated property is hereditary and saturated, and it is not all of V since it does not contain $s(e_1)$. It must therefore be empty. □

A vertex $v \in V$ is *non-wandering* when there is a finite path μ in G such that $v = s(\mu) = r(\mu)$. We denote by NW the set of non-wandering vertexes in V. The *non-wandering subgraph* of G is the subgraph G^{NW} consisting of the vertexes NW and the edges emitted from any of its elements. It follows that G^{NW} is *strongly connected* in the sense that for any pair $v, w \in NW$ there is a finite path μ in G such that $s(\mu) = v$ and $r(\mu) = w$:

Lemma 2.2 *NW is a (possibly empty) hereditary subset of V and the graph G^{NW} is strongly connected.*

Proof When $v \in NW$ there is an infinite path in G which visits v infinitely often. So when $e \in E$ and $s(e) \in NW$ the cofinality of G ensures that there is a path μ in G connecting $r(e)$ to $s(e)$ by Lemma 2.1. Then $e\mu$ is a loop in G containing $r(e)$, proving that $r(e) \in NW$, and hence that NW is hereditary. The proof that G^{NW} is strongly connected is similar. □

Let $\beta \in \mathbb{R}$. We are here looking for maps (or vectors) $\xi : V \to [0, \infty)$ such that

$$\sum_{w \in V} A_{vw}\xi_w = e^{\beta}\xi_v \tag{2.1}$$

for all $v \in V \backslash V_\infty$ and

$$\sum_{w \in V} A_{vw}\xi_w \leq e^{\beta}\xi_v \tag{2.2}$$

for $v \in V_\infty$. We say then that ξ is *almost β-harmonic* for A. When (2.1) holds for all $v \in V$, and not only for $v \in V \backslash V_\infty$, we say that ξ is *β-harmonic* for A.

Lemma 2.3 *Let H be a non-empty hereditary subset of V and $\eta : H \to [0, \infty)$ a function such that (2.1) holds for all $v \in H \backslash V_\infty$ and (2.2) holds for all $v \in H \cap V_\infty$. There is a unique almost β-harmonic vector ξ such that $\xi_v = \eta_v$ for all $v \in H$.*

Proof Set

$$H_1 = \{v \in V \backslash V_\infty : \ A_{vw} \neq 0 \ \Rightarrow \ w \in H\} \cup H.$$

Then H_1 is hereditary, contains H and there is a unique extension of η to H_1 given by the condition that $e^{\beta}\eta_v = \sum_{w \in V} A_{vw}\eta_w$ for all $v \in H_1 \backslash H$. Continuing by induction we get a sequence $H \subseteq H_1 \subseteq H_2 \subseteq H_3 \subseteq \cdots$ of subsets of V and a unique extension ξ of η to $\bigcup_n H_n$. This completes the proof since $\bigcup_n H_n$ is hereditary and saturated, and hence equal to V. □

Lemma 2.4 *Assume that G contains a sink (i.e. A contains a zero row). It follows that for all $\beta \in \mathbb{R}$ there is a non-zero almost β-harmonic vector, unique up to multiplication by constants.*

Proof Note that a sink s constitutes a hereditary subset in itself. It follows therefore from Lemma 2.3 that an almost β-harmonic vector is determined by its value at s. To show that there exists a non-zero almost β-harmonic vector, set $\eta_s = 1$ and apply Lemma 2.3. □

Lemma 2.5 *Let ξ be a non-zero almost β-harmonic vector. It follows that $\xi_v > 0$ for all $v \in V$.*

Proof The set $\{v \in V : \ \xi_v = 0\}$ is hereditary and saturated. Since ξ is not zero the set is not all of V, and it must therefore be empty. □

We define the matrices $A^n, n = 0, 1, 2, \ldots$, recursively such that $A^0 = I$, where I is the identity matrix,

$$I_{vw} = \begin{cases} 1 & \text{when } v = w \\ 0 & \text{otherwise,} \end{cases}$$

$A^1 = A$, and

$$A^{n+1}_{vw} = \sum_{u \in V} A_{vu} A^n_{uw}$$

when $n \geq 1$. While the entries in A are finite by assumption, this need not be the case for A^n, but since (2.1) and (2.2) imply that

$$\sum_{w \in V} A^n_{vw} \xi_w \leq e^{n\beta} \xi_v \tag{2.3}$$

for all v, n and every almost β-harmonic vector ξ, the following conclusion follows from Lemma 2.5.

Lemma 2.6 *Assume that there is a non-zero almost β-harmonic vector for A. It follows that $A^n_{vw} < \infty$ for all $n \in \mathbb{N}$ and all $v, w \in V$.*

In the rest of the paper we therefore assume that all powers of A are finite. Since Lemma 2.4 contains all the information we seek when there is a sink present, there is also nothing lost by assuming that there are no sinks in G. To summarise we assume in the rest of the paper that

i) G is cofinal,
ii) that there are no sinks in G (equivalently, there are no zero rows in A), and
iii) that $A^n_{vw} < \infty$ for all $n \in \mathbb{N}$ and all $v, w \in V$.

In particular, from now on V_∞ consists of the infinite emitters in G, corresponding to rows in A with infinitely many non-zero entries.

Lemma 2.7 *No vertex $v \in V \backslash NW$ is an infinite emitter, i.e. $V_\infty \subseteq NW$.*

Proof Let $v \in V$ be an infinite emitter. Set

$$A = \{w \in V : \text{ there is a finite path } \mu \text{ in } G \text{ such that } w = s(\mu) \text{ and } r(\mu) = v\} \cup \{v\}.$$

Since $V \backslash A$ is hereditary and saturated, it follows that $A = V$. In particular,

$$r(s^{-1}(v)) \subseteq A,$$

which implies that $v \in NW$. □

Lemma 2.8 *Let $\beta \in \mathbb{R}$. Assume that there are vertexes $v_0, w_0 \in V$ such that*

$$\sum_{n=0}^{\infty} A^n_{v_0 w_0} e^{-n\beta} = \infty.$$

Then $\sum_{n=0}^{\infty} A^n_{v w_0} e^{-n\beta} = \infty$ for all vertexes $v \in V$.

Proof Set

$$\mathscr{C} = \left\{ v \in V : \sum_{n=0}^{\infty} A^n_{v w_0} e^{-n\beta} < \infty \right\}.$$

The equality

$$\sum_{u \in V} A_{vu} \sum_{n=0}^{N} A^n_{u w_0} e^{-n\beta} = e^{\beta} \sum_{n=0}^{N+1} A^n_{v w_0} e^{-n\beta} - e^{\beta} I_{v w_0} \tag{2.4}$$

shows that $\mathscr{C}$ is hereditary and saturated. Since $v_0 \notin \mathscr{C}$ it follows that $\mathscr{C} = \emptyset$. □

When NW is not empty we take an element $v \in NW$ and set

$$\beta_0 = \log \left(\limsup_n \left(A^n_{vv}\right)^{\frac{1}{n}} \right)$$

with the convention that $\log \infty = \infty$. Since G^{NW} is strongly connected by Lemma 2.2 the value β_0 is independent of the choice of vertex $v \in NW$, and in fact

$$\beta_0 = \log \left(\limsup_n \left(A^n_{vw}\right)^{\frac{1}{n}} \right),$$

for all $v, w \in NW$. We see from (2.3) that $A^n_{vv} \xi_v \leq e^{n\beta} \xi_v$ for all n when ξ is an almost β-harmonic vector. Since $\xi_v > 0$ by Lemma 2.5 it follows that there cannot be an almost β-harmonic vector for A unless $\beta \geq \beta_0$. We will therefore also in the following assume that

iv) $\beta_0 = \log \left(\limsup_n \left(A^n_{vv}\right)^{\frac{1}{n}} \right) < \infty$ for all $v \in NW$,

when $NW \neq \emptyset$.

3 The Recurrent Case

In this section we consider the case where $NW \neq \emptyset$ and where

$$\sum_{n=0}^{\infty} A^n_{vv} e^{-n\beta_0} = \infty \tag{3.1}$$

for one and hence all $v \in NW$. We say that A is *recurrent* in this case. The main result will be the following theorem. In the irreducible case, i.e. when G is strongly connected, it is contained in Corollary 2 on page 371 in [9].

Theorem 3.1 *Assume that A is recurrent. There is a non-zero almost β_0-harmonic vector which is unique up to multiplication by scalars, and it is β_0-harmonic for A.*

The proof will require some preparations, some of which will also play a role in the following sections. Let $\beta \geq \beta_0$. If the sum

$$\sum_{n=0}^{\infty} A^n_{uu} e^{-n\beta} \tag{3.2}$$

is finite for some $u \in NW$, it will be finite for all $u \in NW$ because G^{NW} is strongly connected by Lemma 2.2. Since

$$\left\{ v \in V : \sum_{n=0}^{\infty} A^n_{vu} e^{-n\beta} < \infty \right\}$$

is hereditary and saturated, we conclude that if the sum (3.2) is finite for some $u \in NW$, the sums

$$\sum_{n=0}^{\infty} A^n_{vu} e^{-n\beta}, \tag{3.3}$$

where $v \in V, u \in NW$, will all be finite. Since $V_\infty \subseteq NW$ by Lemma 2.7, it follows that the sums (3.3) are all finite when $v \in V$ and $u \in V_\infty$, provided the sum (3.2) is finite for one (and hence all) $u \in NW$.

Lemma 3.2 *Let $k : V_\infty \to [0, \infty)$ be a non-negative function on V_∞ such that*

$$\sum_{u \in V_\infty} \sum_{n=0}^{\infty} e^{-n\beta} A^n_{uu} k_u < \infty. \tag{3.4}$$

It follows that there is a vector $\hat{k}$ defined such that

$$\hat{k}_v = \sum_{u \in V_\infty} \sum_{n=0}^{\infty} e^{-n\beta} A^n_{vu} k_u < \infty \tag{3.5}$$

for all $v \in V$, and that $\hat{k}$ is an almost β-harmonic vector.

Proof Straightforward. □

We say that a non-negative function $k : V_\infty \to [0, \infty)$ is β-*summable* when (3.4) holds. When V_∞ is a finite set and $k \neq 0$, this condition is equivalent to the finiteness of the sum (3.2) for any $u \in NW$.

Lemma 3.3 (Riesz Decomposition) *Let ψ an almost β-harmonic vector. There is a unique pair ϕ, k, where ϕ is a β-harmonic vector and $k : V_\infty \to [0, \infty)$ is β-summable, such that*

$$\psi = \phi + \hat{k}. \tag{3.6}$$

Proof The arguments are standard, cf. e.g. 6.43 on page 170 in [10]. ϕ is defined as the limit

$$\phi_v = \lim_{n \to \infty} \sum_{w \in V} e^{-n\beta} A^n_{vw} \psi_w,$$

while

$$k_u = \psi_u - \sum_{v \in V} e^{-\beta} A_{uv} \psi_v.$$

It is then easy to see that (3.6) holds. For uniqueness, assume that ϕ' is β-harmonic, that $k' : V_\infty \to [0, \infty)$ is β-summable and that $\psi = \phi' + \hat{k}'$. Then

$$\lim_{n \to \infty} \sum_{w \in V} e^{-n\beta} A^n_{vw} \hat{k}'_w = 0$$

for all $v \in V$ and hence $\phi'_v = \lim_{n \to \infty} \sum_{w \in V} e^{-n\beta} A^n_{vw} \psi_w = \phi_v$ for all v. Thus $\phi = \phi'$ and $\hat{k} = \hat{k}'$. It follows that

$$k_v = \hat{k}_v - \sum_{w \in V} e^{-\beta} A_{vw} \hat{k}_w = \psi_v - \sum_{w \in V} e^{-\beta} A_{vw} \psi_w = \hat{k}'_v - \sum_{w \in V} e^{-\beta} A_{vw} \hat{k}'_w = k'_v$$

for all $v \in V_\infty$. □

Corollary 3.4 *Assume that A is recurrent. It follows that all almost β_0-harmonic vectors are β_0-harmonic.*

Proof This follows from Lemma 3.3 since no non-zero function $k : V_\infty \to [0, \infty)$ can be β_0-summable in the recurrent case. □

With Corollary 3.4 in place, the proof of Theorem 3.1 can be copied from the work of Vere-Jones, [9]. We introduce for $v, w \in V$ and $n = 0, 1, 2, \dots$, the numbers $r_{vw}(n)$ such that $r_{vw}(0) = 0,\ r_{vw}(1) = A_{vw}$ and

$$r_{vw}(n+1) = \sum_{u \neq w} A_{vu} r_{uw}(n)$$

when $n \geq 1$.

Lemma 3.5 (Equation (4) in [9].) *Assume $NW \neq \emptyset$ and that $\beta > \beta_0$. Then*

$$\sum_{n=0}^{\infty} A^n_{vw} e^{-n\beta} = I_{vw} + \left(\sum_{n=1}^{\infty} r_{vw}(n) e^{-n\beta} \right) \left(\sum_{n=0}^{\infty} A^n_{ww} e^{-n\beta} \right).$$

for all $v, w \in NW$.

Proof By using the product rule for power series the stated equality follows from the observation that for $n \geq 1$, $A^n_{vw} = \sum_{s=1}^{n} r_{vw}(s) A^{n-s}_{ww}$. □

Lemma 3.6 (Lemma 4.1 in [9].) *Assume $\xi : V \to [0, \infty)$ satisfies that*

$$\sum_{w \in V} A_{vw} \xi_w \leq e^{\beta} \xi_v$$

for all v. Assume $\xi_{v_0} \neq 0$ for some vertex $v_0 \in V$. It follows that

$$\sum_{n=1}^{\infty} r_{vv_0}(n) e^{-n\beta} \leq \frac{\xi_v}{\xi_{v_0}}$$

for all v.

Proof We prove by induction in N that

$$\sum_{n=1}^{N} r_{vv_0}(n) e^{-n\beta} \leq \frac{\xi_v}{\xi_{v_0}} \tag{3.7}$$

for all N and all v. To start the induction note that

$$\xi_v \geq e^{-\beta} \sum_{w \in V} A_{vw} \xi_w \geq e^{-\beta} A_{vv_0} \xi_{v_0} = \xi_{v_0} r_{vv_0}(1) e^{-\beta}.$$

Assume then that (3.7) holds for all v. It follows that

$$\begin{aligned}
\frac{\xi_v}{\xi_{v_0}} &\geq e^{-\beta} \sum_{w \in V} A_{vw} \frac{\xi_w}{\xi_{v_0}} = e^{-\beta} \left(\sum_{w \neq v_0} A_{vw} \frac{\xi_w}{\xi_{v_0}} + A_{vv_0} \right) \\
&\geq e^{-\beta} \sum_{n=1}^{N} \sum_{w \neq v_0} A_{vw} r_{wv_0}(n) e^{-n\beta} + e^{-\beta} A_{vv_0} \\
&= \sum_{n=1}^{N} r_{vv_0}(n+1) e^{-(n+1)\beta} + e^{-\beta} r_{vv_0}(1) = \sum_{n=1}^{N+1} r_{vv_0}(n) e^{-n\beta}.
\end{aligned}$$

□

Proof of Theorem 3.1 In view Corollary 3.4 we must prove the existence and essential uniqueness of a non-zero β_0-harmonic vector. Existence: Fix a vertex $w \in NW$. It follows from Fatou's lemma that

$$\lim_{\beta \downarrow \beta_0} \sum_{n=0}^{\infty} A^n_{ww} e^{-n\beta} = \infty.$$

Since

$$\sum_{n=0}^{\infty} A^n_{ww} e^{-n\beta} = 1 + \left(\sum_{n=1}^{\infty} r_{ww}(n) e^{-n\beta} \right) \left(\sum_{n=0}^{\infty} A^n_{ww} e^{-n\beta} \right).$$

for all $\beta > \beta_0$ by Lemma 3.5, it follows that

$$\lim_{\beta \downarrow \beta_0} \sum_{n=1}^{\infty} r_{ww}(n) e^{-n\beta} = 1.$$

By the monotone convergence theorem this leads to the conclusion that

$$\sum_{n=1}^{\infty} r_{ww}(n) e^{-n\beta_0} = 1. \tag{3.8}$$

Now note that

$$\sum_{u \in V} A_{vu} \left(\sum_{n=1}^{N} r_{uw}(n) e^{-n\beta_0} \right)$$

$$= \sum_{n=1}^{N} \sum_{u \neq w} A_{vu} r_{uw}(n) e^{-n\beta_0} + A_{vw} \sum_{n=1}^{N} r_{ww}(n) e^{-n\beta_0} \tag{3.9}$$

$$= \sum_{n=1}^{N} r_{vw}(n+1) e^{-n\beta_0} + A_{vw} \sum_{n=1}^{N} r_{ww}(n) e^{-n\beta_0}$$

$$= e^{\beta_0} \sum_{n=1}^{N+1} r_{vw}(n) e^{-n\beta_0} + A_{vw} \left(\sum_{n=1}^{N} r_{ww}(n) e^{-n\beta_0} - 1 \right).$$

It follows from (3.9) and (3.8) that

$$\left\{ v \in V : \sum_{n=1}^{\infty} r_{vw}(n) e^{-n\beta_0} < \infty \right\}$$

is both hereditary and saturated, and hence equal to V since G is cofinal. By letting N tend to infinity in (3.9) we see that

$$\xi_v = \sum_{n=1}^{\infty} r_{vw}(n) e^{-n\beta_0}$$

defines a β_0-harmonic vector ξ.

Uniqueness: Fix a vertex w and let ξ' be a non-zero β_0-harmonic vector for A such that $\xi'_w = 1$. We must show that $\xi' = \xi$. It follows from Lemma 3.6 that $\xi'_v \geq \xi_v$ for all $v \in V$. By comparing this to the fact that

$$e^{n\beta_0} = \sum_{v \in V} A^n_{wv} \xi_v = \sum_{v \in V} A^n_{wv} \xi'_v$$

for all $n \in \mathbb{N}$, we conclude that $\xi'_v = \xi_v$ for every vertex $v \in V$ with the property that $A^n_{wv} \neq 0$ for some n. In particular, ξ and ξ' agree on NW since $w \in NW$, and G^{NW} is strongly connected by Lemma 2.2. As NW is also hereditary by the same lemma, it follows from Lemma 2.3 that $\xi' = \xi$. □

3.1 When *NW* Is Finite

By definition of the graph G there is at most one directed edge from one vertex to another. So when NW is finite there are no infinite emitters in NW and hence no infinite emitters at all by Lemma 2.7. In particular, all almost β-harmonic vectors are β-harmonic.

Lemma 3.7 *Assume that NW is non-empty but finite. Then A is recurrent and there are no non-zero β-harmonic vectors for A when $\beta > \beta_0$.*

Proof Note that e^{β_0} is the spectral radius of $A|_{NW}$. It follows from linear algebra that there is a non-zero vector $\psi : NW \to \mathbb{C}$ and a complex number $\lambda \in \mathbb{C}$, $|\lambda| = 1$, such that $\sum_{w \in NW} A^n_{vw}\psi_w = e^{n\beta_0}\lambda^n\psi_v$ for all $n \in \mathbb{N}$ and all $v \in NW$. In particular, the sequence $A^n_{vw}e^{-n\beta_0}$ cannot converge to zero for all $v, w \in NW$. It follows that A must be recurrent. Assume that ξ is a non-zero β-harmonic vector. Let $v \in NW$. Then

$$e^{n\beta}\xi_v = \sum_{w \in NW} A^n_{vw}\xi_w \le K \max_{w \in NW} A^n_{vw},$$

where $K = (\#NW)(\max_{w \in NW} \xi_w)$. There is therefore a vertex $w \in NW$ such that

$$e^{n_i\beta}\xi_v \le KA^{n_i}_{vw}$$

for an increasing sequence $n_1 < n_2 < n_3 < \dots$ of natural numbers. It follows that

$$e^{\beta} = \lim_i \left(e^{n_i\beta}\xi_v\right)^{\frac{1}{n_i}} \le \limsup_n \left(KA^n_{vw}\right)^{\frac{1}{n}} = e^{\beta_0},$$

proving that $\beta \le \beta_0$. □

Corollary 3.8 *Assume that NW is non-empty and finite. It follows that there are no non-zero β-harmonic vectors for A unless $\beta = \beta_0$. There is a non-zero β_0-harmonic vector which unique up to multiplication by scalars.*

Proof The first statement follows from Lemma 3.7, and the second from Lemma 3.7 and Theorem 3.1. □

4 The Transient Case

In this section we consider the non-recurrent cases. Specifically, we assume that

$$\sum_{n=0}^{\infty} A^n_{vw}e^{-n\beta} < \infty \tag{4.1}$$

for all $v, w \in V$, and refer to this as the *transient* case. The following lemma shows that the transient case covers all the non-recurrent cases.

Lemma 4.1 *Assume that $NW = \emptyset$ or that $\sum_{n=0}^{\infty} A^n_{uu}e^{-n\beta} < \infty$ for some $u \in NW$. It follows that (4.1) holds for all $v, w \in V$.*

Proof If (4.1) fails for some v, w, it follows from Lemma 2.8 that $\sum_{n=0}^{\infty} A^n_{ww} e^{-n\beta} = \infty$. In particular, $w \in NW$ and $\beta = \beta_0$. This is a recurrent case, contrary to assumption. □

We say that A is *row-finite* when there are no infinite emitter in G, i.e. when

$$\# \{w \in V : A_{vw} > 0\} < \infty$$

for all $v \in V$.

Theorem 4.2 *Let $\beta \in \mathbb{R}$ and assume that (4.1) holds for all $v, w \in V$.*

a) Assume that $NW = \emptyset$. Then A is row-finite and there is a non-zero β-harmonic vector for A.
b) Assume that NW is non-empty but finite. There are no non-zero almost β-harmonic vectors for A.
c) Assume that NW is infinite. There is a non-zero almost β-harmonic vector if and only if $\beta \geq \beta_0$.

Proof In case a), it follows from Lemma 2.7 that A is row-finite. Case b) follows from Lemma 3.7 and Corollary 3.8. It remains therefore only to show that there is a non-zero almost β-harmonic vector in case a) and c). To this end, fix $v_0 \in V$ and set

$$H_{v_0} = \left\{ w \in V : A^l_{v_0 w} \neq 0 \text{ for some } l \in \mathbb{N} \right\}.$$

Consider a vertex $v \in H_{v_0}$ and choose $l \in \mathbb{N}$ such that $A^l_{v_0 v} \neq 0$. Then

$$A^l_{v_0 v} \sum_{n=0}^{\infty} A^n_{vw} e^{-n\beta} \leq \sum_{n=0}^{\infty} A^{l+n}_{v_0 w} e^{-n\beta} \leq e^{l\beta} \sum_{n=0}^{\infty} A^n_{v_0 w} e^{-n\beta}. \tag{4.2}$$

It follows that

$$\frac{\sum_{n=0}^{\infty} A^n_{vw} e^{-n\beta}}{\sum_{n=0}^{\infty} A^n_{v_0 w} e^{-n\beta}} \leq \frac{e^{l\beta}}{A^l_{v_0 v}} \tag{4.3}$$

for all $w \in H_{v_0}$. Note that H_{v_0} is infinite. When NW is infinite this follows since $NW \subseteq H_{v_0}$ by Lemma 2.1. When $NW = \emptyset$ it follows because there are no sinks by assumption. It follows therefore from (4.3) that there is a sequence $\{w_k\}$ of distinct elements in H_{v_0} such that the limit

$$\eta_v = \lim_{k \to \infty} \frac{\sum_{n=0}^{\infty} A^n_{v w_k} e^{-n\beta}}{\sum_{n=0}^{\infty} A^n_{v_0 w_k} e^{-n\beta}}$$

exists for all $v \in H_{v_0}$. Note that $\eta_{v_0} = 1$. By letting N tend to ∞ in (2.4) we find that

$$\sum_{u \in V} A_{vu} \sum_{n=0}^{\infty} A^n_{uw} e^{-n\beta} = e^{\beta} \sum_{n=0}^{\infty} A^n_{vw} e^{-n\beta} - e^{\beta} I_{vw}. \tag{4.4}$$

It follows from (4.4) that $\sum_{u \in V} A_{vu}\eta_u = e^{\beta}\eta_v$ for all $v \in H_{v_0} \backslash V_\infty$, while Fatou's lemma shows that $\sum_{u \in V} A_{vu}\eta_u \leq e^{\beta}\eta_v$ for all $v \in H_{v_0}$. The existence of a non-zero almost β-harmonic vector for A follows then from Lemma 2.3. □

When G is strongly connected and A is row-finite, c) in Theorem 4.2 is a result of Pruitt, [5]. When A is not row-finite it can happen, also when G is strongly connected, that there are no non-zero β-harmonic vectors for any $\beta \geq \beta_0$ or that they exist for some $\beta \geq \beta_0$ and not for others. See [8] for such examples.

4.1 The Structure of the Positive Eigenvectors

We denote the set of almost β-harmonic vectors for A by $E(A, \beta)$. Assume that $E(A, \beta) \neq 0$. For a given vertex $v_0 \in V$ we set

$$E(A, \beta)_{v_0} = \{\xi \in E(A, \beta) : \ \xi_{v_0} = 1\}.$$

Equipped with the product topology $\mathbb{R}^V$ is a locally convex real vector space, and $E(A, \beta)$ is a closed convex cone in $\mathbb{R}^V$. It follows from Lemma 2.5 that $E(A, \beta)_{v_0}$ is a base for $E(A, \beta)$, and we aim now to show that $E(A, \beta)_{v_0}$ is a compact Choquet simplex[1] and to obtain an integral representation of the elements in $E(A, \beta)_{v_0}$, analogous to the Poisson-Martin integral representation for the harmonic functions of a countable state Markov chain, cf. e.g. Theorem 7.45 in [10]. For this purpose we consider the partial ordering $\geq$ in $E(A, \beta)$ defined such that $\xi \geq \mu \ \Leftrightarrow \ \xi - \mu \in E(A, \beta)$.

Lemma 4.3 *$E(A, \beta)$ is a lattice cone; i.e. every pair of elements $\xi, \eta \in E(A, \beta)$ has a least upper bound $\xi \vee \eta \in E(A, \beta)$ and a greatest lower bound $\xi \wedge \eta \in E(A, \beta)$ for the order $\geq$.*

Proof Given a vector $\psi \in E(A, \beta)$ we let (ψ^h, ψ^p) be the unique pair such that ψ^h is β-harmonic and ψ^p a non-negative vector supported on V_∞ fulfilling that $\psi = \psi^p + \widehat{\psi^p}$, cf. Lemma 10. (For simplicity of exposition we consider ψ^p as a non-negative vector on V with support in V_∞.) Given two vectors $\psi, \varphi : V \to [0, \infty)$

[1]We use [1] as reference for the convexity theory we need. The basic definitions and results regarding Choquet simplices are nicely presented there.

we define $\psi \overset{\bullet}{\wedge} \varphi : V \to [0,\infty)$ such that

$$\left(\psi \overset{\bullet}{\wedge} \varphi\right)_v = \min\{\psi_v, \varphi_v\}.$$

Set now

$$(\xi \wedge \eta)_v = \left(\lim_{n\to\infty} \sum_{w\in V} e^{-n\beta} A^n_{v,w} \left(\xi \overset{\bullet}{\wedge} \eta\right)_w\right) + \sum_{n=0}^{\infty} \sum_{w\in V} e^{-n\beta} A^n_{v,w} \left(\xi^p \overset{\bullet}{\wedge} \eta^p\right)_w .$$

The identity

$$\xi_v - (\xi \wedge \eta)_v =$$
$$\left(\xi^h_v - \lim_{n\to\infty} \sum_{w\in V} e^{-n\beta} A^n_{v,w} \left(\xi \overset{\bullet}{\wedge} \eta\right)_w\right) + \sum_{n=0}^{\infty} \sum_{w\in V} e^{-n\beta} A^n_{v,w} \left(\xi^p_w - \left(\xi^p \overset{\bullet}{\wedge} \eta^p\right)_w\right)$$

shows that $\xi \wedge \eta \leq \xi$ in $E(A,\beta)$. Similarly, $\xi \wedge \eta \leq \eta$, showing that $\xi \wedge \eta$ is a lower bound for ξ and η in $E(A,\beta)$. To show that it is the largest lower bound, assume that $\lambda \leq \xi,\ \lambda \leq \eta$ in $E(A,\beta)$. There is then a vector $\kappa \in E(A,\beta)$ such that $\lambda + \kappa = \xi$. The uniqueness in Lemma 10 implies then that $\xi^h = \lambda^h + \kappa^h$ and $\xi^p = \lambda^p + \kappa^p$, showing that $\lambda^h_v \leq \xi^h_v$ and $\lambda^p_v \leq \xi^p_v$ for all $v \in V$. Similarly, $\lambda^h_v \leq \eta^h_v$ and $\lambda^p_v \leq \eta^p_v$ for all $v \in V$. It follows first that $\lambda_v \leq \left(\xi \overset{\bullet}{\wedge} \eta\right)_v$ and $\lambda^p_v \leq \left(\xi^p \overset{\bullet}{\wedge} \eta^p\right)_v$, and hence that $\lambda^h_v = \lim_{n\to\infty} \sum_{w\in V} e^{-n\beta} A^n_{v,w} \lambda_w \leq \lim_{n\to\infty} \sum_{w\in V} e^{-n\beta} A^n_{v,w} \left(\xi \overset{\bullet}{\wedge} \eta\right)_w$ for all $v \in V$. It follows that $\lambda + \kappa' = \xi \wedge \eta$, where

$$\kappa'_v =$$
$$\left(\lim_{n\to\infty} \sum_{w\in V} e^{-n\beta} A^n_{v,w} \left(\xi \overset{\bullet}{\wedge} \eta\right)_w - \lambda^h_v\right) + \sum_{n=0}^{\infty} \sum_{w\in V} e^{-n\beta} A^n_{v,w} \left(\left(\xi^p \overset{\bullet}{\wedge} \eta^p\right)_w - \lambda^p_w\right).$$

Since $\kappa' \in E(A,\beta)$ this shows that $\lambda \leq \xi \wedge \eta$ and we conclude that $\xi \wedge \eta$ is the greatest lower bound for ξ and η in $E(A,\beta)$. To obtain a least upper bound for ξ and η, set

$$\xi \vee \eta = \xi + \eta - \xi \wedge \eta,$$

which is clearly an upper bound for both ξ and η. To show that it is the least, let $\xi \leq \lambda,\ \eta \leq \lambda$ in $E(A,\beta)$. Then $\xi + \eta \leq (\lambda + \xi) \wedge (\lambda + \eta) = \lambda + (\xi \wedge \eta) \Rightarrow \xi + \eta - \xi \wedge \xi \leq \lambda$ in $E(A,\beta)$.

Fix now a vertex $v_0 \in V$. As in the proof of Theorem 4.2, set

$$H_{v_0} = \left\{ w \in V : \ A^n_{v_0 w} \neq 0 \text{ for some } n \in \mathbb{N} \right\},$$

and note that H_{v_0} is a hereditary set of vertexes. For every $v \in V$ we consider the function $K^\beta_v : H_{v_0} \to [0, \infty)$ defined by

$$K^\beta_v(w) = \frac{\sum_{n=0}^\infty A^n_{vw} e^{-n\beta}}{\sum_{n=0}^\infty A^n_{v_0 w} e^{-n\beta}}.$$

Lemma 4.4 *Let $H \subseteq V$ be a non-empty hereditary subset of vertexes. For each $v \in V$ there is a $m_v \in \mathbb{N}$ such that*

$$A^l_{vw} \neq 0, \ l \geq m_v \ \Rightarrow \ w \in H.$$

Proof Define subsets $H_i \subseteq V$ recursively such that $H_0 = H$ and

$$H_{n+1} = \{ v \in V : \ A_{vw} \neq 0 \ \Rightarrow \ w \in H_n \} \cup H_n.$$

Then $H_0 \subseteq H_1 \subseteq H_2 \subseteq \cdots$ is a sequence of hereditary subsets, and the union $\bigcup_n H_n$ is both hereditary and saturated. It is therefore all of V since G is cofinal by assumption. When $v \in H_k$ we can use $m_v = k$. □

Lemma 4.5 *Let $\beta \in \mathbb{R}$ and assume that $\sum_{n=0}^\infty A^n_{vw} e^{-n\beta} < \infty$ for all $v, w \in V$. For every vertex $v \in V$ there are positive numbers l_v, L_v and a finite set $F_v \subseteq H_{v_0}$ such that $K^\beta_v(w) \leq L_v$ for all $w \in H_{v_0}$ and $0 < l_v \leq K^\beta_v(w)$ for all $w \in H_{v_0} \backslash F_v$.*

Proof Consider first a vertex $v \in H_{v_0}$. There is an $l \in \mathbb{N}$ such that $A^l_{v_0 v} \neq 0$. Set $N_v = \left(A^l_{v_0 v}\right)^{-1} e^{l\beta}$ and note that the calculation (4.3) gives the upper bound

$$K^\beta_v(w) \leq N_v \tag{4.5}$$

for all $w \in H_{v_0}$. Consider then a vertex $v \in V \backslash H_{v_0}$. By Lemma 4.4 there is an $m_v \in \mathbb{N}$ such that every path in G of length m_v emitted from v terminates in H_{v_0}. Let Γ denote the set of finite paths μ in G starting at v and terminating in H_{v_0}, and such that $r(\mu)$ is the only vertex in μ which is in H_{v_0}. Then $|\mu| \leq m_v$ for all $\mu \in \Gamma$. Now note that

$$V_\infty \subseteq NW \subseteq H_{v_0},$$

where the first inclusion comes from Lemma 2.7 and the second follows from Lemma 2.1. Since $V_\infty \subseteq H_{v_0}$ and $v \notin H_{v_0}$, the set Γ has only finitely many elements. For every $\mu = e_1 e_2 \cdots e_{|\mu|} \in \Gamma$, set

$$W(\mu) = A_{s(e_1)r(e_1)} A_{s(e_2)r(e_2)} \cdots A_{s(e_{|\mu|})r(e_{|\mu|})} e^{-|\mu|\beta}.$$

Then

$$\begin{aligned}\sum_{n=0}^{\infty} A^n_{vw} e^{-n\beta} &= \sum_{\mu \in \Gamma} W(\mu) \sum_{n \geq |\mu|} A^{n-|\mu|}_{r(\mu)w} e^{-(n-|\mu|)\beta} \\ &= \sum_{\mu \in \Gamma} W(\mu)\, K^{\beta}_{r(\mu)}(w) \left(\sum_{n=0}^{\infty} A^n_{v_0 w} e^{-n\beta} \right) \\ &\leq \left(\sum_{n=0}^{\infty} A^n_{v_0 w} e^{-n\beta} \right) \sum_{\mu \in \Gamma} W(\mu)\, N_{r(\mu)}\end{aligned} \tag{4.6}$$

for all $w \in H_{v_0}$. It follows that we can use $L_v = N_v$ when $v \in H_{v_0}$ and $L_v = \sum_{\mu \in \Gamma} W(\mu)\, N_{r(\mu)}$, otherwise.

To establish the existence of l_v and F_v, assume for a contradiction that for all $\epsilon > 0$ there are infinitely many elements $w \in H_{v_0}$ such that

$$K^{\beta}_v(w) \leq \epsilon.$$

We can then construct a sequence $\{w_k\}$ of distinct elements in H_{v_0} such that

$$K^{\beta}_v(w_k) \leq \frac{1}{k} \tag{4.7}$$

for all k. The calculation (4.4) shows that

$$\sum_{u \in V} A_{v'u} K^{\beta}_u(w_k) = e^{\beta} K^{\beta}_{v'}(w_k) - e^{\beta} \left(\sum_{n=0}^{\infty} A^n_{v_0 w_k} e^{-n\beta} \right)^{-1} I_{v' w_k} \tag{4.8}$$

for all $v' \in V$ and all $k \in \mathbb{N}$. It follows from (4.8) that a condensation point $\xi = (\xi_u)_{u \in V}$ in $\prod_{u \in V} [0, L_u]$ of the sequence

$$\left(K^{\beta}_u(w_k) \right)_{u \in V}, \ k \in \mathbb{N},$$

is an almost β-harmonic vector for A with $\xi_{v_0} = 1$. But (4.7) implies that $\xi_v = 0$ which is impossible by Lemma 2.5. This contradiction shows that there must be an $l_v > 0$ and a finite set $F_v \subseteq H_{v_0}$ such that $l_v \leq K^{\beta}_v(w)$ for all $w \in H_{v_0} \backslash F_v$. $\square$

It follows from Lemma 4.5 that K^{β}_v is a bounded function on H_{v_0}, i.e. $K^{\beta}_v \in l^{\infty}(H_{v_0})$ for all $v \in V$. We denote by 1_w the characteristic function of an element $w \in H_{v_0}$. Let $\mathscr{A}_{\beta}$ be the C^*-subalgebra of $l^{\infty}(H_{v_0})$ generated by K^{β}_v, $v \in V$, and the functions 1_w, $w \in H_{v_0}$, and let $\mathscr{B}_{\beta}$ be the image of $\mathscr{A}_{\beta}$ in the quotient algebra

$$l^{\infty}(H_{v_0}) / c_0 (H_{v_0} \backslash V_{\infty}).$$

Here $c_0(H_{v_0} \backslash V_\infty)$ denotes the ideal in $l^\infty(H_{v_0})$ consisting of the elements $f : H_{v_0} \to \mathbb{C}$ with the property that for all $\epsilon > 0$ there is a finite subset $F \subseteq H_{v_0} \backslash V_\infty$ such that $|f(w)| \leq \epsilon \ \forall w \in H_{v_0} \backslash F$. In particular, $f(V_\infty) = \{0\}$ when $V_\infty \neq \emptyset$ and $f \in c_0(H_{v_0} \backslash V_\infty)$.

Note that it follows from Lemma 4.5 that for every $v \in V$ there is a finite subset $F_v \subseteq H_{v_0}$ such that

$$K_v^\beta + \sum_{w \in F_v} 1_w$$

is invertible in $l^\infty(H_{v_0})$. Thus $\mathscr{A}_\beta$ and $\mathscr{B}_\beta$ are both unital C^*-algebras. Since they are also separable, the set X_β of characters of $\mathscr{B}_\beta$ is a compact metric space and $\mathscr{B}_\beta$ can be identified with $C(X_\beta)$ via the Gelfand transform. Since evaluation at a vertex $w \in V_\infty$ annihilates $c_0(H_{v_0} \backslash V_\infty)$, each element of V_∞ gives rise to character on $\mathscr{B}_\beta$ and hence an element of X_β. It follows that there is a canonical inclusion

$$V_\infty \subseteq X_\beta. \tag{4.9}$$

For each $v \in V$ the function $w \mapsto K_v^\beta(w)$ is an element of $\mathscr{A}_\beta$ and its image in $\mathscr{B}_\beta$ is a continuous function on X_β which we also denote by K_v^β. Let $M(X_\beta)$ denote the set of Borel probability measures on X_β. We consider $M(X_\beta)$ as a compact convex set in the weak*-topology obtained by considering the measures as elements of the dual of $C(X_\beta)$.

Theorem 4.6 *Let $\beta \in \mathbb{R}$. Assume that $NW = \emptyset$ or that NW is infinite. Assume also that $\sum_{n=0}^\infty A_{vw}^n e^{-n\beta} < \infty$ for all $v, w \in V$. Then $E(A, \beta)_{v_0}$ is a non-empty compact metrizable Choquet simplex and there is a continuous affine surjection $I : M(X_\beta) \to E(A, \beta)_{v_0}$ defined such that*

$$I(m)_v = \int_{X_\beta} K_v^\beta \, dm.$$

Proof It follows from Theorem 4.2 that $E(A, \beta)_{v_0} \neq \emptyset$. Since $E(A, \beta)_{v_0}$ is a base of $E(A, \beta)$, which is a lattice cone by Lemma 4.3, to conclude that $E(A, \beta)_{v_0}$ is a compact Choquet simplex we need only show that $E(A, \beta)_{v_0}$ is compact in $\mathbb{R}^V$, cf. e.g. [1]. Note that $I : M(X_\beta) \to \mathbb{R}^V$ is continuous by definition of the topologies. It suffices therefore to show that

$$I\left(M(X_\beta)\right) = E(A, \beta)_{v_0}. \tag{4.10}$$

Let $x \in X_\beta$. If $x \in V_\infty$, we find that

$$K_v^\beta(x) = \frac{\sum_{n=0}^\infty A_{vx}^n e^{-n\beta}}{\sum_{n=0}^\infty A_{v_0 x}^n e^{-n\beta}}$$

and it follows from (4.4) that

$$\sum_{u\in V} A_{vu} K_u^\beta(x) \leq e^\beta K_v^\beta(x) \tag{4.11}$$

for all $v \in V$ with equality when $v \neq x$. To show that (4.11) also holds when $x \in X_\beta \backslash V_\infty$, note first that point-evaluations at points in H_{v_0} constitute a dense subset of the character space of $\mathscr{A}_\beta$. It follows that there is a sequence $\{u_k\} \subseteq H_{v_0}$ such that $K_v^\beta(x) = \lim_{k\to\infty} K_v^\beta(u_k)$ for all $v \in V$. Since $\mathscr{B}_\beta$ is the quotient of $\mathscr{A}_\beta$ by the ideal $c_0(H_{v_0}\backslash V_\infty)$, the sequence $\{u_k\}$ must eventually leave every finite subset of $H_{v_0}\backslash V_\infty$, and since $x \notin V_\infty$ it must also eventually leave every finite subset of V_∞. That is, $\lim_{k\to\infty} u_k = \infty$ in the sense that for any finite set F of vertexes there is an $N \in \mathbb{N}$ such that $u_k \notin F \ \forall k \geq N$. It follows then from (4.4) that (4.11) holds for all $v \in V$ with equality when $v \notin V_\infty$. This shows that the inequality

$$\sum_{u\in V} A_{vu} K_u^\beta \leq e^\beta K_v^\beta$$

holds point-wise on X_β for all $v \in V$, and that equality holds globally on X_β when $v \notin V_\infty$. It follows therefore by integration that $I(m) \in E(A,\beta)_{v_0}$ for all $m \in M(X_\beta)$.

To obtain (4.10) it remains to show that $E(A,\beta)_{v_0} \subseteq I(M(X_\beta))$. For this we modify the argument from the proof of Theorem 4.1 in [7]. Let $\xi \in E(A,\beta)_{v_0}$. Fix an element $x \in X_\beta$ and let $\{w_k\}$ be a sequence of elements in H_{v_0} such that $\lim_{k\to\infty} K_v^\beta(w_k) = K_v^\beta(x)$ for all $v \in V$. Observe that since $K_v^\beta(x) > 0$ by Lemma 2.5 it follows that

$$\lim_{n\to\infty} nK_v^\beta(w_n) = \infty \tag{4.12}$$

for all $v \in V$. For each $n \in \mathbb{N}$, set

$$\xi_v^n = \min\left\{\xi_v,\ nK_v^\beta(w_n)\right\}.$$

Then

$$\lim_{m\to\infty} e^{-m\beta} \sum_{w\in V} A_{vw}^m \xi_w^n = 0 \tag{4.13}$$

for all n, v. To see this note that

$$\left(\sum_{j=0}^\infty A^j_{v_0 w_n} e^{-j\beta}\right) e^{-m\beta} \sum_{w\in V} A^m_{vw}\xi^n_w \leq \left(\sum_{j=0}^\infty A^j_{v_0 w_n} e^{-j\beta}\right) ne^{-m\beta} \sum_{w\in V} A^m_{vw} K^\beta_w(w_n)$$

$$= ne^{-m\beta} \sum_{w\in V} A^m_{vw} \sum_{j=0}^\infty A^j_{ww_n} e^{-j\beta} = n \sum_{j\geq m} A^j_{vw_n} e^{-j\beta}.$$

Hence (4.13) follows because $\sum_{j=0}^{\infty} A^j_{vw_n} e^{-j\beta} < \infty$. Set

$$k_n(v) = \xi^n_v - e^{-\beta} \sum_{w \in V} A_{vw} \xi^n_w .$$

We claim that $k_n \geq 0$. To see this observe first that it follows from (4.8) that $e^{-\beta} \sum_{w \in V} A_{vw} K^\beta_w(w_n) \leq K^\beta_v(w_n)$. Combined with $e^{-\beta} \sum_{w \in V} A_{vw} \xi_w \leq \xi_v$ this implies that

$$e^{-\beta} \sum_{w \in V} A_{vw} \xi^n_w \leq \min \left\{ \xi_v, \ nK^\beta_v(w_n) \right\} = \xi^n_v ,$$

proving the claim. Since

$$\sum_{l=0}^{m} e^{-l\beta} \sum_{w \in V} A^l_{vw} k_n(w) = \xi^n_v - e^{-(m+1)\beta} \sum_{w \in V} A^{m+1}_{vw} \xi^n_w ,$$

it follows from (4.13) that

$$\xi^n_v = \sum_{l=0}^{\infty} e^{-l\beta} \sum_{w \in V} A^l_{vw} k_n(w) = \sum_{w \in H_{v_0}} K^\beta_v(w) h_n(w) \tag{4.14}$$

when $v \in H_{v_0}$, where $h_n(w) = \sum_{l=0}^{\infty} e^{-l\beta} A^l_{v_0 w} k_n(w)$. In particular, it follows from (4.14) that

$$\sum_{w \in H_{v_0}} h_n(w) = \sum_{w \in H_{v_0}} K^\beta_{v_0}(w) h_n(w) = \xi^n_{v_0} \leq \xi_{v_0} = 1 .$$

We can therefore define a positive linear functional μ_n of norm ≤ 1 on $\mathscr{A}_\beta$ such that

$$\mu_n(g) = \sum_{w \in H_{v_0}} g(w) h_n(w) .$$

By compactness of the unit ball in the dual space of $\mathscr{A}_\beta$ there is a strictly increasing sequence $\{n_l\}$ in $\mathbb{N}$ and a positive linear functional μ on $\mathscr{A}_\beta$ such that

$$\mu(g) = \lim_{l \to \infty} \mu_{n_l}(g)$$

for all $g \in \mathscr{A}_\beta$. Since $\lim_{l \to \infty} \xi^{n_l}_v = \xi_v$ by (4.12), it follows from (4.14) that

$$\mu\left(K^\beta_v\right) = \xi_v \tag{4.15}$$

for all $v \in H_{v_0}$. For any fixed $w \in H_{v_0}$ we have that $\xi_w^{n_l} = \xi_w$ for all large l, and hence also that $k_{n_l}(w) = 0$ for all large l when $w \in H_{v_0} \backslash V_\infty$. It follows that $\lim_{l \to \infty} \mu_{n_l}(1_w) = 0$ for all $w \in H_{v_0} \backslash V_\infty$, which shows that μ factors through $\mathscr{B}_\beta$. It follows therefore from (4.15) and the Riesz representation theorem that there is a Borel probability measure m on X_β such that $I(m)_v = \xi_v$ for all $v \in H_{v_0}$. By Lemma 2.3 this implies that $I(m) = \xi$. □

When A is sub-stochastic, irreducible and $\beta = 0$, it follows from the abstract characterisation given in Theorem 7.13 of [10] that the spectrum of $\mathscr{A}_\beta$ is the Martin compactification of the associated Markov chain, cf. Definition 7.17 in [10], while X_β is the Martin boundary when A is also row-finite (has finite range in the sense of [10]). When A is not row-finite X_β consists of the Martin boundary and the vertexes V_∞.

Let δ_x denote the Dirac measure at a point $x \in X_\beta$. Then

$$I(\delta_x)_v = K_v^\beta(x)$$

for all $v \in V$. Thus I takes the extreme points in $M(X_\beta)$ to elements of the form $v \mapsto K_v^\beta(x)$ for some $x \in X_\beta$. By definition of X_β, when $x \in X_\beta \backslash V_\infty$, there is a sequence $\{w_k\}$ of distinct elements in H_{v_0} such that

$$\lim_{k \to \infty} K_v^\beta(w_k) = K_v^\beta(x).$$

Recall now that under a continuous affine surjection between compact convex sets, the pre-image of an extremal point is a closed face and therefore contains an extremal point. In this way we obtain from Theorem 4.6 the following description of the extreme points in $E(A,\beta)_{v_0}$.

Corollary 4.7 *Let $\beta \in \mathbb{R}$. Assume that $NW = \emptyset$ or that NW is infinite. Assume also that $\sum_{n=0}^\infty A_{vw}^n e^{-n\beta} < \infty$ for all $v, w \in V$. Let ξ be an extremal point of $E(A,\beta)_{v_0}$. There is a sequence $\{w_k\}$ of distinct elements in H_{v_0} such that*

$$\xi_v = \lim_{k \to \infty} \frac{\sum_{n=0}^\infty A_{vw_k}^n e^{-n\beta}}{\sum_{n=0}^\infty A_{v_0 w_k}^n e^{-n\beta}} \tag{4.16}$$

for all $v \in V$, or there is a vertex $w \in H_{v_0} \cap V_\infty$ such that

$$\xi_v = \frac{\sum_{n=0}^\infty A_{vw}^n e^{-n\beta}}{\sum_{n=0}^\infty A_{v_0 w}^n e^{-n\beta}} \tag{4.17}$$

for all $v \in V$.

Every infinite emitter $w \in V_\infty$ gives rise to an extremal element in $E(A,\beta)_{v_0}$ via the formula (4.17) (in the transient case we consider here). This follows from the uniqueness part in the Riesz decomposition lemma, Lemma 3.3. The question

about which sequences of vertexes $\{w_k\}$ give rise to extremal β-harmonic by the formula (4.16) is generally much more difficult to answer. But at least we shall show below that the vertexes in $\{w_k\}$ can be chosen to be the vertexes in an infinite path in G without repeated vertexes. It follows from this that there are cases where A is not row-finite and where there are no non-zero β-harmonic vectors, for any β, because there are no paths in G of this sort. This can occur also when all our standing assumptions hold and G is strongly connected. See Example 2.9 in [6].

Let $\partial E(A,\beta)_{v_0}$ be the extreme boundary of the Choquet simplex $E(A,\beta)_{v_0}$. By identifying an element $x \in X_\beta$ with the corresponding Dirac measure $\delta_x \in M(X_\beta)$, we have an inclusion $X_\beta \subseteq M(X_\beta)$, and we set

$$\partial X_\beta = X_\beta \cap I^{-1}(\partial E(A,\beta)_{v_0}),$$

which is a Borel subset of X_β, cf. Theorem 4.1.11 in [1]. Thus ∂X_β consists of the elements $x \in X_\beta$ for which $v \mapsto K_v^\beta(x)$ is extremal in $E(A,\beta)_{v_0}$, and it corresponds to the minimal Martin boundary, [10], when X_β is identified as a Martin boundary.

Lemma 4.8 *$V_\infty \subseteq \partial X_\beta$ and I is injective on ∂X_β.*

Proof Let $u \in V_\infty$. Then

$$I(u)_v = K_v^\beta(u) = \widehat{k(u)}_v,$$

where $k(u) : V_\infty \to [0,\infty)$ is the function

$$k(u)(w) = \begin{cases} \left(\sum_{n=0}^\infty A^n_{v_0u} e^{-n\beta}\right)^{-1} & \text{when } w = u,\\ 0 & \text{otherwise.}\end{cases}$$

It follows therefore from the uniqueness part of the statement in Lemma 3.3 that $V_\infty \subseteq \partial X_\beta$ and that I is injective on V_∞. Furthermore, $\partial E(A,\beta)_{v_0}$ is the disjoint union

$$\partial E(A,\beta)_{v_0} = \partial H(A,\beta)_{v_0} \sqcup \{I(u) : \ u \in V_\infty\},$$

where $\partial H(A,\beta)_{v_0}$ is the set of extremal β-harmonic elements of $E(A,\beta)_{v_0}$. It suffices therefore now to show that I is injective on $\partial X_\beta \cap I^{-1}(\partial H(A,\beta)_{v_0})$. Since $I(u)$ is not β-harmonic when $u \in V_\infty$, it follows that

$$X_\beta \cap I^{-1}(\partial H(A,\beta)_{v_0}) \subseteq X_\beta \backslash V_\infty.$$

As $\mathscr{B}_\beta$ is generated by the image of the functions $K_v^\beta, v \in V$, and $1_u, u \in V_\infty$, it follows that I is injective on $X_\beta \backslash V_\infty$ and therefore also on $\partial X_\beta \cap I^{-1}(\partial H(A,\beta)_{v_0})$. □

Let $M(\partial X_\beta)$ denote the subset of $M(X_\beta)$ consisting of the elements m of $M(X_\beta)$ with $m(\partial X_\beta) = 1$.

Theorem 4.9 *The map $I : M(\partial X_\beta) \to E(A,\beta)_{v_0}$ is an affine bijection.*

Proof Surjectivity: Let $\psi \in E(A,\beta)_{v_0}$. Since $E(A,\beta)_{v_0}$ is a metrizable Choquet simplex there is a unique Borel probability measure ν on $\partial E(A,\beta)_{v_0}$ such that

$$\psi = \int_{\partial E(A,\beta)_{v_0}} z\, \nu(z),$$

in the sense that

$$a(\psi) = \int_{\partial E(A,\beta)_{v_0}} a(z)\, \nu(z),$$

for all continuous and affine functions a on $E(A,\beta)_{v_0}$, cf. Theorem 4.1.15 in [1]. By Lemma 4.8 we can define a Borel probability measure m on ∂X_β such that $m(B) = \nu(I(B))$. Extending m to a Borel probability measure on X_β such that $m\left(X_\beta \backslash \partial X_\beta\right) = 0$, we can consider $I(m)$. For each $v \in V$, the evaluation map $\mathrm{ev}_v(\phi) = \phi_v$ is continuous and affine on $E(A,\beta)_{v_0}$ so we find that

$$\begin{aligned} I(m)_v &= \int_{\partial X_\beta} K_v^\beta(y)\, dm(y) = \int_{\partial E(A,\beta)_{v_0}} K_v^\beta(I^{-1}(z))\, d\nu(z) \\ &= \int_{\partial E(A,\beta)_{v_0}} \mathrm{ev}_v\,(z)\;\, d\nu(z) = \mathrm{ev}_v(\psi) = \psi_v. \end{aligned}$$

Injectivity: Assume $m_1, m_2 \in M(\partial X_\beta)$ and that $I(m_1) = I(m_2)$. Using Lemma 4.8 again it follows that there are Borel probability measures ν_i on $\partial E(A,\beta)_{v_0}$ such that $m_i = \nu_i \circ I$ on ∂X_β, $i = 1, 2$. Note that

$$\begin{aligned} I(m_i)_v &= \int_{\partial X_\beta} K_v^\beta(y)\, dm_i(y) = \int_{\partial E(A,\beta)_{v_0}} K_v^\beta(I^{-1}(z))\, d\nu_i(z) \\ &= \int_{\partial E(A,\beta)_{v_0}} \mathrm{ev}_v(z)\, d\nu_i(z) = \mathrm{ev}_v\left(\int_{\partial E(A,\beta)_{v_0}} z\, d\nu_i(z)\right) \end{aligned} \tag{4.18}$$

for all $v \in V, i = 1, 2$. As $I(m_1) = I(m_2)$ it follows that

$$\int_{\partial E(A,\beta)_{v_0}} z\, d\nu_1(z) = \int_{\partial E(A,\beta)_{v_0}} z\, d\nu_2(z).$$

Since $\partial E(A,\beta)_{v_0}$ is a Choquet simplex this implies that $\nu_1 = \nu_2$, and hence $m_1 = m_2$.

□

Corollary 4.10 $\partial X_\beta \backslash V_\infty = \left\{x \in \partial X_\beta : \sum_{u \in V} A_{vu} K_u^\beta(x) = e^\beta K_v^\beta(x)\ \forall v \in V\right\}.$

Proof Let $x \in \partial X_\beta \backslash V_\infty$. Then $I(x) \in \partial E(A, \beta)_{v_0}$ and it follows from the Riesz decomposition, Lemma 3.3, that $I(x)$ is either β-harmonic or equal to $I(w)$ for some $w \in V_\infty$. Since $x \notin V_\infty$ and I is injective on $M(\partial X_\beta)$ by Theorem 4.9, it follows that $I(x)$ is β-harmonic, i.e. $\sum_{u \in V} A_{vu} K_u^\beta(x) = e^\beta K_v^\beta(x)$ for all $v \in V$. □

It follows from Theorem 4.9 and Corollary 4.10 that the map I of Theorem 4.6 restricts to an affine bijection between the β-harmonic elements of $E(A, \beta)_{v_0}$ and the Borel probability measures m on X_β with the property that $m\left(\partial X_\beta \backslash V_\infty\right) = 1$.

4.2 An Improvement

The purpose with this section is to use results from the theory of Markov chains to improve the description of the extremal β-harmonic vectors given in Corollary 4.7.

Let $P(V)$ denote the set

$$P(V) = \left\{(v_i)_{i=1}^\infty \in V^{\mathbb{N}} : A_{v_i v_{i+1}} > 0, i = 1, 2, 3, \cdots \right\}.$$

Since there are no multiple edges in G the elements of $P(V)$ are the infinite paths in G. We consider $P(V)$ as a complete metric space whose topology is generated by the cylinder sets

$$C(v_1 v_2 \cdots v_n) = \left\{(x_i)_{i=1}^\infty \in P(V) : x_i = v_i, \ i = 1, 2, \cdots, n\right\}.$$

Lemma 4.11 *Let ψ be a non-zero β-harmonic vector. There is a Borel measure m_ψ on $P(V)$ such that*

$$m_\psi\left(C(v_1 v_2 \cdots v_n)\right) = A_{v_1 v_2} A_{v_2 v_3} \cdots A_{v_{n-1} v_n} e^{-(n-1)\beta} \psi_{v_n}$$

for every cylinder set $C(v_1 v_2 \cdots v_n)$.

Proof The matrix

$$B_{vw} = e^{-\beta} \psi_v^{-1} A_{vw} \psi_w \tag{4.19}$$

is stochastic. It follows then from Theorem 1.12 in [10] that there is a Borel probability measure m_v on $C(v)$ for each $v \in V$ such that

$$m_v\left(C(v v_2 \cdots v_n)\right) = \psi_v^{-1} A_{v v_2} A_{v_2 v_3} \cdots A_{v_{n-1} v_n} e^{-(n-1)\beta} \psi_{v_n}. \tag{4.20}$$

Define m_ψ such that

$$m_\psi(B) = \sum_{v \in V} \psi_v m_v\left(C(v) \cap B\right).$$

□

Theorem 4.12 *Assume that* $\sum_{n=0}^{\infty} A^n_{vw} e^{-n\beta} < \infty$ *for all* $v, w \in V$. *Let* ψ *be an extremal non-zero* β*-harmonic vector with* $\psi_{v_0} = 1$. *There is an infinite path* $t = (t_i)_{i=1}^{\infty}$ *in* $P(V)$ *such that* $t_1 = v_0$, $i \neq j \Rightarrow t_i \neq t_j$, *and*

$$\psi_v = \lim_{k\to\infty} \frac{\sum_{n=0}^{\infty} A^n_{vt_k} e^{-n\beta}}{\sum_{n=0}^{\infty} A^n_{v_0 t_k} e^{-n\beta}} \tag{4.21}$$

for all $v \in V$.

Proof By construction the measure m_v on $C(v)$ given by (4.20) is the measure $\mathbf{Pr}_v$ from Theorem 1.12 in [10] and the measure P_x, with $x = v$, from (2.2) in [7], coming from the stochastic matrix B from (4.19). Since ψ is extremal in $E(A, \beta)_{v_0}$ the constant vector 1 is minimal harmonic for the stochastic matrix B restricted to H_{v_0}. It follows therefore from Theorem 5.1 in [7] that with respect to the measure m_{v_0} almost all elements in

$$\{(x_i) \in P(V) : \ x_1 = v_0\} \tag{4.22}$$

have the property that

$$\lim_{i\to\infty} \frac{\sum_{n=0}^{\infty} B^n_{vx_i}}{\sum_{n=0}^{\infty} B^n_{v_0 x_i}} = 1 \tag{4.23}$$

for all $v \in H_{v_0}$. Let $w \in H_{v_0}$. Since $\sum_{n=0}^{\infty} B^n_{ww} = \sum_{n=0}^{\infty} A^n_{ww} e^{-n\beta} < \infty$, it follows from Theorems 3.2 and 3.4 in [10] that the set

$$\{(x_i) \in P(V) : \ x_1 = v_0, \ x_i = w \text{ for at most finitely many } i\}$$

has full m_{v_0}-measure. Since H_{v_0} is countable it follows that so has

$$\bigcap_{w \in H_{v_0}} \{(x_i) \in P(V) : \ x_1 = v_0, \ x_i = w \text{ for at most finitely many } i\},$$

which is the same set as $\{(x_i) \in P(V) : \ x_1 = v_0, \ \lim_{i\to\infty} x_i = \infty\}$. Hence the set of elements (x_i) from (4.22) for which (4.23) holds for all $v \in H_{v_0}$ and at the same have the property that $\lim_{i\to\infty} x_i = \infty$, is also a set of full m_{v_0}-measure. Since

$$\frac{\sum_{n=0}^{\infty} B^n_{vx_i}}{\sum_{n=0}^{\infty} B^n_{v_0 x_i}} = \psi_v^{-1} \frac{\sum_{n=0}^{\infty} A^n_{vx_i} e^{-n\beta}}{\sum_{n=0}^{\infty} A^n_{v_0 x_i} e^{-n\beta}},$$

it follows that there is an element (t'_i) from (4.22) such that $\lim_{i\to\infty} t'_i = \infty$ and

$$\lim_{i\to\infty} \frac{\sum_{n=0}^{\infty} A^n_{vt'_i} e^{-n\beta}}{\sum_{n=0}^{\infty} A^n_{v_0 t'_i} e^{-n\beta}} = \psi_v \tag{4.24}$$

for all $v \in H_{v_0}$. It follows then from the uniqueness part of the statement in Lemma 2.3 that (4.24) holds for all $v \in V$. Finally, since $\lim_{i\to\infty} t'_i = \infty$ it is easy to construct from t' a path $(t_i) \in P(V)$ such that $i \neq j \Rightarrow t_i \neq t_j$, and such that there is a sequence $n_1 < n_2 < n_3 < \cdots$ in $\mathbb{N}$ with $t_i = t'_{n_i}$ for all i. The sequence (t_i) has the stated properties. □

In summary we have found that in the transient case the extremal elements of $E(A,\beta)_{v_0}$ consist of the vectors arising from an element $w \in H_{v_0} \cap V_\infty$ by the formula (4.17) when they are not β-harmonic, and are given by an infinite path $(t_i) \in P(V)$ such that $i \neq j \Rightarrow t_i \neq t_j$ via the formula (4.21) when they are β-harmonic.

References

1. O. Bratteli and D.W. Robinson, *Operator Algebras and Quantum Statistical Mechanics I + II*, Texts and Monographs in Physics, Springer Verlag, New York, Heidelberg, Berlin, 1979 and 1981.
2. T.M. Carlsen and N. Larsen, *Partial actions and KMS states on relative graph C^*-algebras*, arXiv:1311.0912.
3. J. Christensen and K. Thomsen, *Finite digraphs and KMS states*, J. Math. Anal. Appl. **433** (2016), 1626–1646.
4. A. an Huef, M. Laca, I. Raeburn and A. Sims, *KMS states on the C^*-algebras of reducible graphs*, Ergodic Th. & Dynam. Syst., to appear.
5. W.E. Pruitt, *Eigenvalues of non-negative matrices*, Ann. Math. Statist. **35** (1964), 1797–1800.
6. S. Ruette, *On the Vere-Jones classification and existence of maximal measure for countable topological Markov chains*, Pac. J. Math. **209** (2003), 365–380.
7. S.A. Sawyer, *Martin boundary and random walks*, Harmonic functions on trees and buildings (New York, 1995), 17–44, Contemp. Math. 206, Amer. Math. Soc., Providence, RI, 1997.
8. K. Thomsen, *KMS weights on graph C^*-algebras*, arXiv:1409.3702
9. D. Vere-Jones, *Ergodic properties of non-negative matrices I*, Pacific J. Math. **22** (1967), 361–386.
10. W. Woess, *Denumerable Markov Chains*, EMS Textbooks in Mathematics, 2009.

Classification of Graph Algebras: A Selective Survey

Mark Tomforde

Abstract This survey reports on current progress of programs to classify graph C^*-algebras and Leavitt path algebras up to Morita equivalence using K-theory. Beginning with an overview and some history, we trace the development of the classification of simple and nonsimple graph C^*-algebras and state theorems summarizing the current status of these efforts. We then discuss the much more nascent efforts to classify Leavitt path algebras, and we describe the current status of these efforts as well as outline current impediments that must be solved for this classification program to progress. In particular, we give two specific open problems that must be addressed in order to identify the correct K-theoretic invariant for classification of simple Leavitt path algebras, and we discuss the significance of various possible outcomes to these open problems.

1 Introduction

In 1976 Elliott proved a result, now known as Elliott's theorem, which states that direct limits of semisimple finite-dimensional algebras may be classified up to isomorphism by the dimension group (later identified with the scaled, ordered K_0-group) of the algebra. Elliott's theorem implies that for the class of AF-algebras (i.e., C^*-algebraic direct limits of finite-dimensional C^*-algebras) the scaled, ordered topological K_0-group is a complete isomorphism invariant, and it also implies that in the class of ultramatricial algebras (i.e., algebraic direct limits of direct sums of semisimple finite-dimensional algebras over a fixed field) the scaled, ordered algebraic K_0-group is a complete isomorphism invariant. (In this case the algebraic K_0-group and topological K_0-group coincide.) Based on this result, Elliott boldly proposed that many additional classes of (C^*-)algebras can be classified up to isomorphism by K-theoretic invariants, and he famously formulated what is now called Elliott's conjecture: "All separable nuclear simple C^*-algebras are classified up to isomorphism by K-theoretic invariants." Work on this conjecture has been referred to as the Elliott program, and over the past four decades there have been

M. Tomforde (✉)
Department of Mathematics, University of Houston, Houston, TX 77204-3008, USA
e-mail: tomforde@math.uh.edu

T.M. Carlsen et al. (eds.), *Operator Algebras and Applications*, Abel Symposia 12,
DOI 10.1007/978-3-319-39286-8_14

numerous contributions made by several mathematicians, not the least of which are due to Elliott himself. Very recently, work of Tikuisis, White, and Winter has completed the final steps for classifying all unital, separable, simple, and nuclear C^*-algebras of finite nuclear dimension which satisfy the UCT [21].

The established work on the Elliott program is vast—indeed, papers on the subject total thousands of pages and even a survey of all the accomplishments over the past four decades would most likely require a document the size of a book if it wished to be comprehensive. (See the book [16] for an introductory survey of the Elliott program with an emphasis on providing a technical overview of the Kirchberg-Phillips classification theorem, and see the papers [9, 17] for a summary of the accomplishments in the Elliott program over the past 15 years.)

In the discussions here we wish to focus on two natural extensions of the Elliott program:

Extension 1: Go beyond the simple C^*-algebras and attempt to classify certain nonsimple nuclear C^*-algebras using K-theory.

Extension 2: Step outside of the class of C^*-algebras and attempt to classify certain simple algebras using (algebraic) K-theory.

Of course there is little to no hope these two extensions can be accomplished for all nuclear C^*-algebras or all simple algebras, so one major component of each program is to identify classes of nonsimple C^*-algebras and simple algebras that are amenable to classification. Another major component of each program is determining exactly what K-theoretic data is needed for the classifying invariant.

Significant progress for Extension 1 has been made for the class of graph C^*-algebras, while progress for Extension 2 has been more difficult, but had some stunning successes for the class of Leavitt path algebras. These two classes, which we collectively refer to as *graph algebras*, will be the focus of this survey.

Readers who have no prior experience with graph algebras may initially (and incorrectly!) believe these classes are fairly small and specialized. However, keep in mind that every AF-algebra is Morita equivalent to a graph C^*-algebra [5] and every ultramatricial algebra over $\mathbb{C}$ is Morita equivalent to a Leavitt path algebra. Thus the graph C^*-algebras and Leavitt path algebras are generalizations of the AF-algebras and ultramatricial algebras to which Elliott's theorem from 1976 applies. As such, they are very suitable classes to explore at the beginnings of these programs.

Moreover, every Kirchberg algebra with free K_1-group is Morita equivalent to a graph C^*-algebra [20], so the class of graph C^*-algebras also generalizes many of the simple C^*-algebras to which the Kirchberg-Phillips classification theorem applies [14]. Consequently, the graph C^*-algebras comprise a large class containing several nonsimple C^*-algebras as well as many simple C^*-algebras of both AF and purely infinite type. At the same time, the graph C^*algebras are not too large to escape classification. The proposed invariant, called the filtered (or sometimes "filtrated") K-theory, is a natural generalization of the invariant used for simple C^*-algebras, and contains the collection of all ordered K_0-groups and K_1-groups of subquotients of the C^*-algebra taking into account all the natural transformations among them. While this seems like the obvious choice for the invariant, Meyer and

Nest have constructed two separable, purely infinite C^*-algebras in the bootstrap class (each with a primitive ideal space having four points) that have the same filtered K-theory but are not Morita equivalent, thus demonstrating that filtered K-theory is inadequate to classify general nonsimple nuclear C^*-algebras. This example also means that restricting to the class of graph algebras is not an artificial choice, but done out of necessity to obtain working theorems. The counterexamples of Meyer and Nest lie outside the class of graph C^*-algebras, and the current working conjecture is that filtered K-theory suffices to classify graph C^*-algebras.

While a promising candidate for the classifying invariant of graph C^*-algebras has been identified, and many initial cases have been established successfully (see [6] for a taxonomy), classification results for Leavitt path algebras have been more piecemeal, and the correct invariant for classification is still uncertain. In this survey, we give a summary of the current status of classification results for simple and nonsimple graph C^*-algebras, and we describe how these results have guided initial work on classifying the simple Leavitt path algebras. We also outline the existing classification results for simple Leavitt path algebras, and describe the current search for a complete Morita equivalence invariant in terms of K-theory. We state two important open questions currently facing the classification program for simple Leavitt path algebras, and we also discuss the implications of various answers to these two open questions.

The "selective survey" of the title refers to the fact that our attention will be primarily be concentrated on the "geometric" classifications described in terms of moves on the graphs. We will discuss the role that dynamical systems have played in these geometric classifications, and outline how dynamics results have been applied to graph algebras. We will omit proofs in favor of focusing on the big picture, but do our best to explain the key ideas used to obtain results. In addition, to avoid getting bogged down in too many technicalities, throughout this survey we shall restrict our attention to classification up to Morita equivalence (eschewing any mention of results for classification up to isomorphism).

In addition to giving an update on current research, the author believes that the narrative, like many instances of mathematical investigation, provides an interesting story of the twists and turns that have occurred as several researchers have contributed to an area of investigation.

2 Preliminaries

In this section we establish notation and state some standard definitions. To begin, we mention that we shall allow infinite graphs, but work under the standing hypothesis that all our graphs are countable.

Definition 2.1 A *graph* (E^0, E^1, r, s) consists of a countable set E^0 of vertices, a countable set E^1 of edges, and maps $r : E^1 \to E^0$ and $s : E^1 \to E^0$ identifying the range and source of each edge. A graph is *finite* if both the vertex set E^0 and the edge set E^1 are finite.

Let $E := (E^0, E^1, r, s)$ be a graph. We say that a vertex $v \in E^0$ is a *sink* if $s^{-1}(v) = \emptyset$, and we say that a vertex $v \in E^0$ is an *infinite emitter* if $|s^{-1}(v)| = \infty$. A *singular vertex* is a vertex that is either a sink or an infinite emitter, and we denote the set of singular vertices by E^0_{sing}. We also let $E^0_{\text{reg}} := E^0 \setminus E^0_{\text{sing}}$, and refer to the elements of E^0_{reg} as *regular vertices*; i.e., a vertex $v \in E^0$ is a regular vertex if and only if $0 < |s^{-1}(v)| < \infty$.

2.1 Definitions of Graph Algebras

Graph C^*-algebras were introduced in the 1990s, motivated by (and generalizing) earlier constructions, such as the Cuntz algebras and the Cuntz-Krieger algebras.

Definition 2.2 (The Graph C^*-Algebra) If E is a graph, the *graph C^*-algebra* $C^*(E)$ is the universal C^*-algebra generated by mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries with mutually orthogonal ranges $\{s_e : e \in E^1\}$ satisfying

1. $s_e^* s_e = p_{r(e)}$ for all $e \in E^1$
2. $s_e s_e^* \leq p_{s(e)}$ for all $e \in E^1$
3. $p_v = \sum_{\{e \in E^1 : s(e) = v\}} s_e s_e^*$ for all $v \in E^0_{\text{reg}}$.

Based on the success of graph C^*-algebras, in 2005 algebraists were inspired to define algebraic analogues, which are called Leavitt path algebras [1].

Definition 2.3 (The Leavitt Path Algebra) Let E be a graph, and let K be a field. We let $(E^1)^*$ denote the set of formal symbols $\{e^* : e \in E^1\}$. The *Leavitt path algebra of E with coefficients in K*, denoted $L_K(E)$, is the free associative K-algebra generated by a set $\{v : v \in E^0\}$ of pairwise orthogonal idempotents, together with a set $\{e, e^* : e \in E^1\}$ of elements, modulo the ideal generated by the following relations:

1. $s(e)e = er(e) = e$ for all $e \in E^1$
2. $r(e)e^* = e^* s(e) = e^*$ for all $e \in E^1$
3. $e^* f = \delta_{e,f}\, r(e)$ for all $e, f \in E^1$
4. $v = \displaystyle\sum_{\{e \in E^1 : s(e) = v\}} ee^*$ whenever $v \in E^0_{\text{reg}}$.

As with the graph C^*-algebras, the Leavitt path algebras include many well-known classes of algebras and have been studied intensely in the algebra community since their introduction. The interplay between these two classes of "graph algebras" has been extensive and mutually beneficial—graph C^*-algebra results have helped to

guide the development of Leavitt path algebras by suggesting what results are true and in what direction investigations should be focused, and Leavitt path algebras have given a better understanding of graph C^*-algebras by helping to identify those aspects of $C^*(E)$ that are algebraic, rather than C^*-algebraic, in nature. Moreover, results from each class have had nontrivial applications to the other, and the work of researchers from each side has guided discovery for the other. Indeed, nearly every theorem from each class seems to have a corresponding theorem in the other. For example, the graph-theoretic conditions on E for which $C^*(E)$ is a simple algebra (respectively, an AF-algebra, a purely infinite simple algebra, an exchange ring, a finite-dimensional algebra) in the category of C^*-algebras are precisely the same graph-theoretic conditions on E for which $L_K(E)$ is a simple algebra (respectively, an ultramatricial algebra, a purely infinite simple algebra, an exchange ring, a finite-dimensional algebra) in the category of K-algebras. The exact reason for these similar properties is a bit of a mystery—the graph C^*-algebra and Leavitt path algebra theorems are proven using different techniques, and the theorems for one class do not seem to imply the theorems for the other in any obvious way. It has been suggested that there may exist some kind of "Rosetta Stone" that would allow for translating and deducing one set of theorems from the other, but currently such a Rosetta Stone remains elusive.

We mention that when the underlying field of the Leavitt path is the complex numbers, then $L_{\mathbb{C}}(E)$ is isomorphic to a dense $*$-subalgebra of $C^*(E)$. However, this alone is not enough to account for the similar results, since it is possible for dense $*$-subalgebras to have considerably different properties and structure from the ambient C^*-algebra.

It is also noteworthy that the field plays little role in most theorems for the Leavitt path algebras, and properties of $L_K(E)$ are frequently obtained entirely in terms of the graph E with no dependence on the field K. We will see in the final section of this survey that classification by K-theory is one of the few examples where this is not true, and the underlying field will be important in our theorems and in the invariants used for classification of simple Leavitt path algebras.

2.2 Computation of K-Groups

We shall use the notation $K_n(A)$ for the n^{th} topological K-group of a C^*-algebra A, and we shall use the notation $K_n^{\text{alg}}(R)$ for the n^{th} algebraic K-group of a ring R. Due to a phenomenon called Bott periodicity, for any C^*-algebra A we have $K_n(A) \cong K_{n+2}(A)$ for all $n \in \mathbb{Z}$. Thus for C^*-algebras, all K-group information is contained in the K_0-group and K_1-group, and these are typically the only K-groups mentioned. For rings (and Leavitt path algebras) there is no periodicity and all the algebraic K-groups may be distinct.

Computation of the topological K-groups of a graph C^*-algebra and the algebraic K-groups of a Leavitt path algebra is described in the following definition and proposition.

Definition 2.4 If $E = (E^0, E^1, r, s)$ is a graph, we define the vertex matrix A_E to be the (possible infinite) square matrix indexed by the vertex set E^0, and for each $v, w \in E^0$ the entry $A_E(v, w)$ is equal to the number of edges in E from v to w. Note that the entries of A_E take values in $\{0, 1, 2, \ldots, \infty\}$.

In the following proposition, for a set X and an abelian group G, we use the notation $G^X := \bigoplus_{x \in X} G$.

Proposition 2.5 (Computation of K-Groups for Graph Algebras) *Let E be a graph and decompose the vertices of E as $E^0 = E^0_{\mathrm{reg}} \sqcup E^0_{\mathrm{sing}}$, and with respect to this decomposition write the vertex matrix of E as*

$$A_E = \begin{pmatrix} B_E & C_E \\ * & * \end{pmatrix}$$

where B_E and C_E have entries in $\mathbb{Z}$ and the $*$'s have entries in $\mathbb{Z} \cup \{\infty\}$. For each $v \in E^0$, let $\delta_v \in \mathbb{Z}^{E^0}$ denote the vector with 1 in the v^{th} position and 0's elsewhere, and for $x \in \mathbb{Z}^{E^0}$ let $[x]$ denote the equivalence class of x in $\operatorname{coker}\left(\begin{pmatrix} I - B_E^t \\ -C_E^t \end{pmatrix} : \mathbb{Z}^{E^0_{\mathrm{reg}}} \to \mathbb{Z}^{E^0}\right)$.

(a) The topological K-theory of the graph C^*-algebra may be calculated as follows: We have

$$K_0(C^*(E)) \cong \operatorname{coker}\left(\begin{pmatrix} I - B_E^t \\ -C_E^t \end{pmatrix} : \mathbb{Z}^{E^0_{\mathrm{reg}}} \to \mathbb{Z}^{E^0}\right)$$

via an isomorphism that takes $[p_v]_0 \mapsto [\delta_v]$ and takes the positive cone of $K_0(C^*(E))$ to the cone of $\operatorname{coker}\left(\begin{pmatrix} I - B_E^t \\ -C_E^t \end{pmatrix} : \mathbb{Z}^{E^0_{\mathrm{reg}}} \to \mathbb{Z}^{E^0}\right)$ *generated by the elements*

$$\{[p_v - \sum_{e \in F} s_e s_e^*] : v \in E^0, F \subseteq s^{-1}(v), \text{ and } F \text{ finite}\},$$

and we have

$$K_1(C^*(E)) \cong \ker\left(\begin{pmatrix} I - B_E^t \\ -C_E^t \end{pmatrix} : \mathbb{Z}^{E^0_{\mathrm{reg}}} \to \mathbb{Z}^{E^0}\right).$$

(b) If K is any field, then the algebraic K-theory of the Leavitt path algebra $L_K(E)$ may be calculated as follows: We have

$$K_0^{\mathrm{alg}}(L_K(E)) \cong \operatorname{coker}\left(\begin{pmatrix} I - B_E^t \\ -C_E^t \end{pmatrix} : \mathbb{Z}^{E^0_{\mathrm{reg}}} \to \mathbb{Z}^{E^0}\right)$$

via an isomorphism that takes $[v]_0 \mapsto [\delta_v]$ *and takes the positive cone of* $K_0^{\mathrm{alg}}(L_K(E))$ *to the cone of* $\operatorname{coker}\left(\begin{pmatrix} I - B_E^t \\ -C_E^t \end{pmatrix} : \mathbb{Z}^{E^0_{\mathrm{reg}}} \to \mathbb{Z}^{E^0}\right)$ *generated by the elements*

$$\{[v - \sum_{e \in F} ee^*] : v \in E^0, F \subseteq s^{-1}(v),\ \textit{and } F \textit{ finite}\},$$

and we have

$$K_1^{\mathrm{alg}}(L_K(E)) \cong \ker\left(\begin{pmatrix} I - B_E^t \\ -C_E^t \end{pmatrix} : \mathbb{Z}^{E^0_{\mathrm{reg}}} \to \mathbb{Z}^{E^0}\right)$$
$$\oplus \operatorname{coker}\left(\begin{pmatrix} I - B_E^t \\ -C_E^t \end{pmatrix} : (K_1^{\mathrm{alg}}(K))^{E^0_{\mathrm{reg}}} \to (K_1^{\mathrm{alg}}(K))^{E^0}\right)$$

with $(K_1^{\mathrm{alg}}(K), +) \cong (K^\times, \cdot)$. *Moreover, there is a long exact sequence*

$$K_n^{\mathrm{alg}}(K)^{E^0_{\mathrm{reg}}} \xrightarrow{\begin{pmatrix} I - B_E^t \\ -C_E^t \end{pmatrix}} K_n^{\mathrm{alg}}(K)^{E^0} \longrightarrow K_n^{\mathrm{alg}}(L_K(E)) \longrightarrow K_{n-1}^{\mathrm{alg}}(K)^{E^0_{\mathrm{reg}}}$$

for $n \in \mathbb{Z}$.

Remark When E has no singular vertices (which occurs, for example, whenever E is finite with no sinks), then one may make the substitution $\begin{pmatrix} I - B_E^t \\ -C_E^t \end{pmatrix} = I - A_E^t$ in all the above expressions.

Throughout this survey we shall be concerned with classification of C^*-algebras and algebras up to Morita equivalence. If A is an algebra (or C^*-algebra) from a given class, and an object $\mathrm{I}(A)$ is assigned to A, we call the assignment a *Morita equivalence invariant* for the class if

$$A \text{ Morita equivalent to } B \implies \mathrm{I}(A) = \mathrm{I}(B) \quad \text{for all } A, B \text{ in the class}$$

and we call the assignment a *complete Morita equivalence invariant* for the class if

$$A \text{ Morita equivalent to } B \iff \mathrm{I}(A) = \mathrm{I}(B) \quad \text{for all } A, B \text{ in the class.}$$

3 Classification of Simple and Nonsimple Graph C^*-Algebras

All graph C^*-algebras are nuclear and in the bootstrap class to which the UCT applies. Furthermore, the standing assumption that our graphs are countable ensures the associated graph C^*-algebras are separable. Simple graph C^*-algebras are either

AF (and classified by Elliott's theorem) or purely infinite (and classified by the Kirchberg-Phillips classification theorem). Consequently, when E is a graph and $C^*(E)$ is simple, the pair

$$((K_0(C^*(E)), K_0(C^*(E))^+), K_1(C^*(E)))$$

consisting of the (pre-)ordered K_0-group together with the K_1-group is a complete Morita equivalence invariant for $C^*(E)$. By looking at the ordering on the K_0-group, we can tell whether $C^*(E)$ is purely infinite or AF: If $K_0(C^*(E))^+ = K_0(C^*(E))$, then $C^*(E)$ is purely infinite and the invariant reduces to the pair $(K_0(C^*(E)), K_1(C^*(E))$; while if $K_0(C^*(E))^+ \subsetneq K_0(C^*(E))$, then $C^*(E)$ is AF, $K_1(C^*(E)) = 0$, and the ordered K_0-group

$$(K_0(C^*(E)), K_0(C^*(E))^+)$$

classifies $C^*(E)$ up to Morita equivalence.

While the classification of simple graph C^*-algebras is a special case of the existing classification theorems for simple nuclear C^*-algebras (specifically Elliott's theorem and the Kirchberg-Phillips classification theorem), rather than merely considering the graph C^*-algebra result as a corollary, it perhaps better to take a historical view and think of simple graph C^*-algebras as the test cases that provided intermediate steps successively leading to more general theorems for classifying simple nuclear C^*-algebras. Indeed, the classification of AF-algebras up to Morita equivalence given by Elliott's theorem, which initiated the entire classification program, is tantamount to classifying C^*-algebras of graphs with no cycles (since the two classes coincide up to Morita equivalence). Likewise, the purely infinite simple graph C^*-algebras (especially particular subclasses) provided important initial steps leading to the Kirchberg-Phillips Classification theorem.

For example, Cuntz first calculated the K-theory of the Cuntz algebra $\mathcal{O}_n$ (which is the C^*-algebra of a graph with one vertex and n edges) in the paper [3], showing that $K_0(\mathcal{O}_n) \cong \mathbb{Z}/n\mathbb{Z}$. This implies a very specific case of the Kirchberg-Phillips theorem for the Cuntz algebras: $\mathcal{O}_m$ is Morita equivalent to $\mathcal{O}_n \iff K_0(\mathcal{O}_m) \cong K_0(\mathcal{O}_n) \iff m = n$.

Likewise, the classification of simple Cuntz-Krieger algebras was an important early step in the classification program. (The Cuntz-Krieger algebras are precisely the C^*-algebras of finite graphs with no sinks or sources.) Simple Cuntz-Krieger algebras are purely infinite and classified up to Morita equivalence by their K_0-group. (The K_1-group turns out to be redundant, because the K_1-group of a Cuntz-Krieger algebra is isomorphic to the free part of the K_0-group.) The groundwork for this classification was laid by Cuntz and Krieger [4], who recognized the connection with dynamics, and the final portions of the classification were later established by Rørdam (using an important lemma outlined by Cuntz in a talk) [15]. We will return to this result, its proof, and the connection with dynamics in the next section.

In addition to providing stepping stones toward more general classification theorems for simple nuclear C^*-algebras, the graph C^*-algebras have also provided

a class for exploring classification of nonsimple C^*-algebras. In a graph C^*-algebra with finitely many ideals, each quotient and each ideal is Morita equivalent to a graph C^*-algebra, so any graph C^*-algebra with finitely many ideals may be built up from the simple graph C^*-algebras by taking extensions a finite number of times.

The invariant used to classify a nonsimple graph C^*-algebra $C^*(E)$ is called the filtered (or sometimes "filtrated") K-theory and denoted $\mathrm{FK}^+_X(C^*(E))$, where X is the primitive ideal space of $C^*(E)$. The filtered K-theory contains the collection of all ordered K_0-groups and K_1-groups of subquotients of the C^*-algebra, taking into account all the natural transformations among them (see [6, §1] for a precise definition). When $C^*(E)$ has a single (nonzero) ideal, the filtered K-theory is simply the six-term exact sequence in K-theory (including the ordering on all K_0-groups) determined by the unique ideal. Eilers and the author [8] proved that the filtered K-theory (i.e., the six-term exact sequence) is a complete Morita equivalence invariant for the class of graph C^*-algebras with one ideal. This kicked off a flurry activity in which several researchers established classification results for graph C^*-algebras with multiple ideals. A summary of these results and description of the status quo for this program can be found in the survey [6]—in particular, the authors there describe how we have a complete classification of graph C^*-algebras with a primitive ideal space having three or fewer points, and for graph C^*-algebras whose primitive ideal space has four points 103 of the 125 cases have been solved, leaving less than one fifth of the cases open. In addition, as we will discuss at the end of Sect. 4, Eilers, Restorff, and Ruiz recently announced that they have recently shown that filtered K-theory is a complete Morita equivalence invariant for all unital graph C^*-algebras.

4 In Search of Techniques to Classify Algebras: Shift Spaces, Flow Equivalence, and Moves on Graphs

Many of the techniques used to establish classification theorems for C^*-algebras have no hope of going through for general algebras. Indeed, the classification proofs frequently make use of the C^*-algebra structure (e.g., the completeness is used frequently to take limits). As one undertakes a classification of algebras, the first step is to begin with simple algebras. In addition, it is sensible to restrict initial attention to algebras that are somehow similar to the C^*-algebras for which the classification program had early successes using more modest methods. The graph C^*-algebras (particularly the Cuntz-Krieger algebras) are such a class, and hence their algebraic analogues, the Leavitt path algebras (particularly Leavitt path algebras of finite graphs), arise as natural candidates for attempts at classification.

Simple Leavitt path algebras exhibit a dichotomy similar to simple graph C^*-algebras: A simple Leavitt path algebra is either ultramatricial (if the graph has no cycles) or purely infinite (if the graph has a cycle). In the ultramatricial case, Elliott's theorem applies and the Leavitt path algebra is classified by its ordered K_0-group. In the purely infinite case, we seek an algebraic analogue of the Kirchberg-Phillips

classification theorem, and therefore we look at how the classification was obtained for early investigations into special cases—more specifically, we shall carefully examine the classification of simple Cuntz-Krieger algebras.

The Cuntz-Krieger algebras correspond to C^*-algebras of finite graphs with no sinks and no sources, and the simplicity of the Cuntz-Krieger algebras corresponds to the graph being strongly connected (i.e., for each pair of vertices v and w there is a path from v to w and a path from w to v) and not a single cycle. Let $E = (E^0, E^1, r, s)$ be a finite graph with no sinks and no sources. One may define the (two-sided) shift space

$$X_E := \{\dots e_{-2}e_{-1}.e_0e_1e_2\dots \ : \ e_i \in E^1 \text{ and } r(e_i) = s(e_{i+1}) \text{ for all } i \in \mathbb{Z}\}$$

consisting of all bi-infinite paths in the graph, together with the shift map $\sigma : X_E \to X_E$ given by $\sigma(\dots e_{-2}e_{-1}.e_0e_1e_2\dots) = \dots e_{-1}e_0.e_1e_2e_3\dots$. Cuntz and Krieger observed a connection between the (characterizations of) flow equivalence of this dynamical system and the Morita equivalence class of the C^*-algebra associated with the graph.

If E and F are finite graphs with no sinks and no sources, the shift spaces X_E and X_F are *flow equivalent* if their suspension flows are homeomorphic via a homeomorphism that carries orbits to orbits and preserves each orbit's orientation. A precise definition of flow equivalence may be found in [12, §13.6], but as we shall see shortly, for the purposes of this survey one does not need to understand flow equivalence so much as note that it is equivalent to other conditions.

Franks gave an algebraic characterization of flow equivalence for strongly connected graphs: If E and F are strongly connected finite graphs, then X_E and X_F are flow equivalent if and only if

$$\operatorname{coker}(1 - A_E^t) = \operatorname{coker}(1 - A_F^t) \ \text{ and } \ \operatorname{sgn}(\det(1 - A_E^t)) = \operatorname{sgn}(\det(1 - A_F^t).$$

Here A_E is the vertex matrix of the graph E, and $\operatorname{sgn}(\det(1 - A_E^t))$ is the sign of the number $\det(1 - A_E^t)$; i.e., the value $+$, $-$, or 0.

In addition, Parry and Sullivan gave a different characterization of flow equivalence based on "moves"; i.e., operations that may be performed on the graph and which preserve flow equivalence of the associated shift space. Parry and Sullivan needed three moves for their characterization, which are named as follows:

Move (O): Outsplitting **Move (I):** Insplitting **Move (R):** Reduction

Precise definitions of these moves can be found in [18, §3], but for the purposes of this survey any readers unfamiliar with the moves may be better served by informal descriptions: *Outsplitting* allows one to partition the outgoing edges of a vertex into nonempty sets and "split" the vertex and incoming edges so that each set of outgoing edges from the partition now emits from its own vertex. *Insplitting* allows one to partition the ingoing edges of a vertex into nonempty sets and "split" the vertex and outgoing edges so that each set of ingoing edges from the partition now enters its

own vertex. *Reduction* allows one to "collapse" certain vertices that have a single edge going from one vertex to the other.

For each move there is also an *inverse move*, so that if a Move X is applied to the graph E to obtain the graph E', we say E is obtained by performing the inverse of Move X to E'. Although we will not need their names, for the reader's edification we mention that the inverse of outsplitting is called *outamalgamation*, the inverse of insplitting is called *inamalgamation*, and the inverse of reduction is called *delay*.

Parry and Sullivan proved that if E and F are strongly connected graphs, then X_E and X_F are flow equivalent if and only if the graph E may turned into the graph F by finitely many applications of Moves (O), (I), (R), and their inverses.

Combining Franks result with the result of Parry and Sullivan, we thus obtain the following:

Theorem 4.1 (Franks, Parry and Sullivan) *Let E and F be strongly connected finite graphs. Then the following are equivalent:*

(1) The shift spaces X_E and X_F are flow equivalent.
(2) $\operatorname{coker}(1 - A_E^t) = \operatorname{coker}(1 - A_F^t)$ *and* $\operatorname{sgn}(\det(1 - A_E^t)) = \operatorname{sgn}(\det(1 - A_F^t))$.
(3) The graph E may turned into the graph F by finitely many applications of Moves (O), (I), (R), and their inverses.

Cuntz and Krieger had multiple insights to recognize the relationship of flow equivalence with the Morita equivalence of Cuntz-Krieger algebras. First, after calculating the K-groups of a Cuntz-Krieger algebra, Cuntz and Krieger observed that the K_0-group coincides with the group $\operatorname{coker}(1 - A_E^t)$ appearing in the flow equivalence classification. (This group is sometimes called the Bowen-Franks group by dynamicists.) The second important observation of Cuntz and Krieger was that the Moves (O), (I), (R) (and consequently their inverses) preserve Morita equivalence of the associated C^*-algebra.

These observations, combined with Theorem 4.1, imply that if E and F are strongly connected finite graphs with isomorphic K_0-groups and with $\operatorname{sgn}(\det(1 - A_E^t)) = \operatorname{sgn}(\det(1 - A_F^t))$, then E can be transformed into F using a finite number of Moves (O), (I), (R) and their inverses, and consequently the C^*-algebras of E and F are Morita equivalent. (Note that we really only need the equivalence of (2) and (3) in Theorem 4.1, and for the purposes of Cuntz and Krieger's result, we can completely ignore the notion of flow equivalence if we wish, viewing (2) $\iff$ (3) as a purely combinatorial fact about graphs.)

Although Cuntz and Krieger formulated their study of the Cuntz-Krieger algebras in terms of matrices, we wish to use the more modern approach of describing the C^*-algebras in terms of graphs. The Cuntz-Krieger algebras may be thought of as the C^*-algebras of finite graphs with no sinks or sources. If a graph C^*-algebra is simple and purely infinite, then the graph cannot contain a sink, so when we restrict to the simple purely infinite case we automatically have the "no sinks" condition for Cuntz-Krieger algebras. However, we do need a method to deal with sources, and to

accomplish this we introduce a new move:

Move (S): Source Removal

A precise definition of Move (S) can be found in [18, §3], but an informal description is fairly accurate and informative: To perform Move (S) we select a source vertex in the graph and then remove this vertex and all edges beginning at this vertex. As with the other moves, Move (S) preserves Morita equivalence of the associated C^*-algebra. The is also an inverse move called *source addition*. Note that the process of performing Move (S) removes a source, but may create other sources in doing so. Nonetheless, one can easily show that in a finite graph with no sinks, repeated applications of Move (S) will ultimately (and in a finite number of steps) result in a graph with no sources.

The following is a reformulation of Cuntz and Krieger's result in the language of graphs that also takes the presence of sources into account.

Theorem 4.2 (Cuntz and Krieger) *Let E and F be finite graphs for which $C^*(E)$ and $C^*(F)$ are simple and purely infinite. If $K_0(C^*(E)) \cong K_0(C^*(F))$ and*

$$\operatorname{sgn}(\det(1 - A_E^t)) = \operatorname{sgn}(\det(1 - A_F^t)),$$

then $C^*(E)$ is Morita equivalent to $C^*(F)$. Moreover, in this case, the graph E may turned into the graph F by finitely many applications of Moves (S), (O), (I), (R), and their inverses.

Cuntz and Krieger suspected that $\operatorname{sgn}(\det(1 - A_E^t))$ was not a necessary condition for Morita equivalence of the C^*-algebras, but they were unable to remove the hypothesis from their theorem. It was not until 15 years later that Rørdam was able to remove the "sign of the determinant" condition and obtain a complete Morita equivalence invariant for Cuntz-Krieger algebras. To accomplish this, Rørdam used an additional graph move that did not appear in the study of flow equivalence. This move is called the *Cuntz splice*, and because it will be important for us in the remainder of this survey, we shall describe it in greater detail than the other moves.

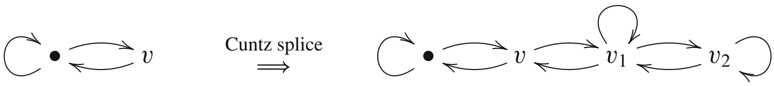
Move (CS): Cuntz Splice

If E is a graph and v is any vertex in E that is the base of two distinct cycles, then Move (CS) is performed by "splicing" on the following additional portion to E:

$$\begin{tikzcd} v \arrow[r, bend left] & v_1 \arrow[l, bend left] \arrow[loop above] \arrow[r, bend left] & v_2 \arrow[l, bend left] \arrow[loop right] \end{tikzcd}$$

Here is an example showing the Cuntz splice performed on a graph with two vertices and three edges to produce a new graph with four vertices and nine edges.

Example 4.3



Unlike the other moves, we shall have no need of an inverse move for the Cuntz splice. The usefulness of the Cuntz splice is due to the following fact: If E is a graph and E^- is obtained by performing a Cuntz splice to a vertex of E, then $K_0(C^*(E)) \cong K_0(C^*(E^-))$ and $\operatorname{sgn}(\det(I - A_E^t)) = -\operatorname{sgn}(\det(I - A_{E^-}^t))$. In other words, the Cuntz splice preserves the K_0-group of the associated C^*-algebra while "flipping" the sign of the determinant. Rørdam's main contribution was to prove that the Cuntz splice preserves Morita equivalence of the associated C^*-algebra. Using this fact, one can start with two purely infinite simple graph C^*-algebras having the same K_0-group. If the signs of the determinants are the same, simply apply Theorem 4.2 to deduce Morita equivalence. If not, apply the Cuntz splice once to one of the graphs to switch the sign of the determinant, and then apply Theorem 4.2 to deduce Morita equivalence. Rørdam phrased his result in terms of the matrix description of Cuntz-Krieger algebras, but we reformulate it here in the modern language of graphs, which also takes the presence of sources into account.

Theorem 4.4 (Rørdam) *Let E and F be finite graphs for which $C^*(E)$ and $C^*(F)$ are simple and purely infinite. Then the following are equivalent:*

(1) $C^*(E)$ *is Morita equivalent to* $C^*(F)$.
(2) $K_0(C^*(E)) \cong K_0(C^*(F))$.
(3) The graph E may turned into the graph F by finitely many applications of Moves (S) (O), (I), (R), and their inverses, and at most one application of Move (CS). Moreover, no applications of Move (CS) are needed if $\operatorname{sgn}(\det(1 - A_E^t)) = \operatorname{sgn}(\det(1 - A_F^t))$*, and exactly one application of Move (CS) is required otherwise.*

Theorem 4.4 shows that $K_0(C^*(E))$ is a complete invariant for Morita equivalence in the class of simple purely infinite C^*-algebras of finite graphs. Even better than that, Theorem 4.4 gives moves on the graph generating the equivalence relation (in analogy to moves for other equivalence relations, such as the Reidemeister moves for the isotopy class of a knot). This allows one to turn the question of Morita equivalence of the C^*-algebras into a combinatorial problem on graphs. For this reason, the moves of Theorem 4.4 are sometimes said to give a "geometric classification" of these graph C^*-algebras.

In addition, for finite graphs whose C^*-algebras are purely infinite and simple, Theorems 4.1 and 4.4 explain the precise relationship between flow equivalence of the shift space and Morita equivalence of the C^*-algebra. In particular, $K_0(C^*(E))$ is a complete invariant for Morita equivalence of $C^*(E)$, and the pair $(K_0(C^*(E)), \operatorname{sgn}(\det(1 - A_E^t)))$ is a complete invariant for flow equivalence of X_E. Consequently, for these graphs the flow equivalence of the shift space is

a finer equivalence relation than Morita equivalence of the C^*-algebra (i.e., X_E flow equivalent to X_F implies that $C^*(E)$ is Morita equivalent to $C^*(F)$, but not conversely in general).

After Rørdam's work and this geometric classification of Cuntz-Krieger algebras, efforts in the classification program progressed in ways that were less "geometric". However, in 2005 Abrams and Aranda Pino introduced Leavitt path algebras. In 2008, approximately 13 years after the geometric classification for simple Cuntz-Krieger algebras was obtained, Abrams, Louly, Pardo, and Smith were inspired to seek a similar geometric classification for simple Leavitt path algebras of finite graphs [2]. We will discuss the Leavitt path algebra classification in Sect. 5, and as with graph C^*-algebras we shall see the sign of the determinant condition is a stumbling block. Unlike the C^*-algebra situation, however, this problem has not been resolved and it is an open question as to whether the sign of the determinant may be removed. We will discuss the ramifications of this question, and the implications of possible answers, in the next section.

With Leavitt path algebra considerations causing attention to be returned to a geometric classification, Sørensen had the novel idea to reconsider the graph C^*-algebras and seek geometric classifications for simple C^*-algebras of infinite graphs [19]. Although the Kirchberg-Phillips theorem, established in 2000, showed that the pair of the K_0-group and K_1-group is a complete Morita equivalence invariant for purely infinite simple graph C^*-algebras, the result is highly non-geometric and does not allow one to establish the Morita equivalence in any concrete or constructive way. Sørensen's key insight was to realize that classification by moves could still be obtained when the graph has a finite number of vertices and an infinite number of edges. In this case, the K-groups of the C^*-algebras are obtained as the cokernel and kernel of a finite rectangular (but not square) matrix indexed by the vertices. In this situation, due to the fact there are infinitely many edges, one cannot define a shift space as before. This is because definition of a shift spaces requires a finite alphabet (i.e., finitely many edges) from which each position in the bi-infinite sequences may be chosen. Despite this, one can still focus on the equivalence of (2) and (3) in Theorem 4.1, considering it as a purely combinatorial fact about graphs, and seek an analogous result for infinite graphs with finitely many vertices. As one would expect, the object $\operatorname{coker}(I - A_E^t) \cong K_0(C^*(E))$ from the finite graph case must now be replaced by the pair $(K_0(C^*(E)), K_1(C^*(E))$. It is less clear what quantity should play the analogous role of $\operatorname{sgn}(\det(I - A_E^t))$, since the matrix involved is not square and hence its determinant does not exist. Surprisingly, Sørensen proved that the sign of the determinant condition simply disappears in the presence of infinitely many edges, and—in what is even better news—this means there is no need for the Cuntz splice. Sørensen's results from [19] may be summarized as follows.

Theorem 4.5 (Sørensen) *Let E and F be graphs with a finite number of vertices and an infinite number of edges and with the property that $C^*(E)$ and $C^*(F)$ are simple. Then the following are equivalent:*

(1) $C^*(E)$ is Morita equivalent to $C^*(F)$.
(2) $K_0(C^*(E)) \cong K_0(C^*(F))$ and $K_1(C^*(E)) \cong K_1(C^*(F))$.
(3) The graph E may turned into the graph F by finitely many applications of Moves (S), (O), (I), (R), their inverses.

The fact that Sørensen's result does not involve the Cuntz splice has two important consequences: First, the moves (S), (O), (I), (R), and their inverses produce explicit Morita equivalences, and consequently one can concretely construct the imprimitivity bimodule linking $C^*(E)$ to $C^*(F)$ by using full corners of the C^*-algebras of the intermediary graphs between E and F. (The Cuntz splice does not produce an explicit Morita equivalence, so this concrete construction cannot be accomplished for the finite graphs in Theorem 4.4 when $\operatorname{sgn}(\det(1 - A_E^t)) \neq \operatorname{sgn}(\det(1 - A_F^t))$.) Second, when seeking a version of Theorem 4.5 for Leavitt path algebras, the sign of the determinant condition is no longer present to cause difficulties.

A graph C^*-algebra $C^*(E)$ is unital precisely when the graph E has a finite number of vertices. Since any unital simple graph C^*-algebra is either Morita equivalent to $\mathbb{C}$ or purely infinite, the combination of Theorems 4.4 and 4.5 gives a complete classification of unital simple graph C^*-algebras up to Morita equivalence.

In May 2015 Eilers, Restorff, Ruiz, and Sørensen posted a preprint [7] to the arXiv in which they extended the geometric classification to all unital graph C^*-algebras of real rank zero. Specifically, they show that filtered K-theory is a complete Morita equivalence invariant for unital graph C^*-algebras of real rank zero, and that when Morita equivalence occurs, one graph may be turned into the other using a finite number of Moves (S), (O), (I), (R), their inverses, and Move (CS). In July 2015, Eilers, Restorff, Ruiz, and Sørensen posted an update to their arXiv entry stating that they are now able to remove the hypothesis of real rank zero and give a geometric classification for all unital graph C^*-algebras. They also stated that the preprint [7] will not be published, and instead a paper with the more general results (containing all results of [7] as special cases) will be written and published in its place (see http://front.math.ucdavis.edu/1505.06773). In talks, Eilers has stated that the filtered K-theory is a complete Morita equivalence invariant for unital graph C^*-algebras and that a geometric classification is possible. However, when the unital graph C^*-algebra is not real rank zero, we must include one additional move besides Moves (S), (O), (I), (R), their inverses, and Move (CS), in order to handle the situation of cycles with no exits. Since this additional move was discovered while Eilers and Restorff were visiting Ruiz at his home institution in Hawai'i, and since the move involves a graphical picture similar to a butterfly, the authors have tentatively called the move "pulelehua"—the Hawaiian word for butterfly.

5 Classification of Leavitt Path Algebras of Finite Graphs

Since the Leavitt path algebras are defined in a manner analogous to the graph C^*-algebras, they are a natural candidate for a class of algebras that may be amenable to classification by K-theory. In addition, the geometric classification of unital graph C^*-algebras, described in terms of moves on the graph, provides a viable approach to classification for Leavitt path algebras.

Abrams, Louly, Pardo, and Smith initiated the classification of Leavitt path algebras in [2]. (A preprint of [2] was posted to the arXiv in 2008, and a published version appeared in 2011.) To begin, they observed that for any graph E and any field K, one has $K_0^{\text{alg}}(L_K(E)) \cong K_0(C^*(E))$ so that the algebraic K_0-group of the Leavitt path algebra agrees with the K_0-group of the graph C^*-algebra and is independent of the field K. In addition, Abrams, Louly, Pardo, and Smith proved that the graph moves (S), (O), (I), (R), and their inverses preserve Morita equivalence of the Leavitt path algebra of the graph. However, they were unable to determine whether or not the Cuntz splice preserves Morita equivalence of the associated Leavitt path algebra. Thus, following the proof strategy established by Cuntz and Krieger (with later contributions by Rørdam), they could not avoid the sign of the determinant condition and were only able to establish sufficient conditions for Morita equivalence. We state their result here.

Theorem 5.1 (Abrams, Louly, Pardo, and Smith) *Let K be any field, and let E and F be finite graphs for which $L_K(E)$ and $L_K(F)$ are simple and purely infinite. If $K_0^{\text{alg}}(L_K(E)) \cong K_0^{\text{alg}}(L_K(F))$ and $\operatorname{sgn}(\det(1-A_E^t)) = \operatorname{sgn}(\det(1-A_F^t))$, then $L_K(E)$ is Morita equivalent to $L_K(F)$. Moreover, in this case, the graph E may turned into the graph F by finitely many applications of Moves (S), (O), (I), (R), and their inverses.*

One noteworthy consequence of this result is that, as with most of the Leavitt path algebra results, the field plays no role in determining the Morita equivalence class of the Leavitt path algebra for these particular types of graphs. Indeed, the invariant sufficient for classification, the pair $(K_0^{\text{alg}}(L_K(E)), \operatorname{sgn}(\det(1-A_E^t)))$, depends only on the graph E and is independent of the field.

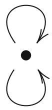
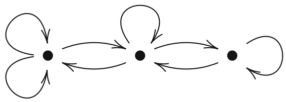
Currently, it is unknown whether the sign of the determinant is a Morita equivalence invariant for Leavitt path algebras. This puts the classification program for simple Leavitt path algebras of finite graphs in a similar state that the classification of simple graph C^*-algebras found itself in during the 15 years period following Cuntz and Krieger's work and prior to Rørdam's contribution of the Cuntz splice. As a result, resolving whether the sign of the determinant can be removed is currently one of the central issues in the classification of simple Leavitt path algebras. This problem is equivalent to determining whether the Cuntz splice preserves Morita equivalence, and thus the open question may be formulated as follows.

Open Question 1 *Let K be a field and let E be a finite graph such* $\det(1-A_E^t) \neq 0$ *and $L_K(E)$ is simple and purely infinite. If E^- denotes the graph obtained by*

performing a Cuntz splice to E, then are the Leavitt path algebras $L_K(E)$ and $L_K(E^-)$ Morita equivalent?

This question has been open since the first preprint of [2] appeared in 2008, and it is currently at the forefront of the classification program for Leavitt path algebras. Many researchers have worked on this problem, but with little to show for their efforts. In fact, we currently cannot answer the question in even elementary special cases.

For example, suppose we take E_2 to be the graph with one vertex and two edges (arguably, the most basic example of the graphs the question is asking about), and let E_2^- be the graph obtained by performing a Cuntz splice at the vertex of E.

E_2  E_2^- 

Then $L_K(E_2)$ is isomorphic to the Leavitt algebra L_2, and we have $K_0^{\mathrm{alg}}(L_K(E_2)) \cong K_0^{\mathrm{alg}}(L_K(E_2^-)) \cong \{0\}$, $\det(1 - A_{E_2}^t) = 1$, and $\det(1 - A_{E_2^-}^t) = -1$. However, it is currently an open question as to whether $L_K(E_2)$ and $L_K(E_2^-)$ are Morita equivalent.

Besides restricting the graph, another approach to finding a more tractable special case of Open Question 1 is to restrict the field, for instance to fix $K = \mathbb{C}$ or $K = \mathbb{Z}_2$. However, no results for this special case have been obtained either. Even combinations of these restrictions (i.e., restricting both the graph and the field) yield open problems—no one knows, for example, whether $L_{\mathbb{C}}(E_2)$ and $L_{\mathbb{C}}(E_2^-)$ are Morita equivalent, or whether $L_{\mathbb{Z}_2}(E_2)$ and $L_{\mathbb{Z}_2}(E_2^-)$ are Morita equivalent.

In July 2015, Johansen and Sørensen announced the preprint [11], which to the author's knowledge contain some of the first concrete results concerning the sign of the determinant condition. Although Leavitt path algebras are defined over fields, as noted by the author in [22], for any graph E and any commutative ring R it is possible to construct a Leavitt path algebra $L_R(E)$ with coefficients in R. Johansen and Sørensen proved that if we choose the coefficients to be the ring $\mathbb{Z}$, then $L_{\mathbb{Z}}(E_2)$ is not $*$-isomorphic to $L_{\mathbb{Z}}(E_2^-)$. (This is contrasted with the graph C^*-algebra situation, where $C^*(E_2)$ is $*$-isomorphic to $C^*(E_2^-)$.) Consequently, the Cuntz splice does not preserve $*$-isomorphism of Leavitt path algebras over the ring $\mathbb{Z}$. What this means—if anything—for Morita equivalence (instead of $*$-isomorphism) of Leavitt path algebras over fields (instead of rings), is yet unclear. But at the very least, Johansen and Sørensen's result shows us that not all Cuntz splice results for graph C^*-algebras will generalize to algebras over commutative rings, and this raises the potential for some unexpected phenomena with Leavitt path algebras over fields. More importantly, up to this point a preponderance of researchers' efforts have been spent trying to prove that the Cuntz splice does preserve Morita equivalence of Leavitt path algebras (over fields). Johansen and Sørensen's result suggests that perhaps we should be spending more time trying to establish the negative.

The lack of an answer to Open Question 1 is currently a major stumbling block in the classification program for Leavitt path algebras. The fact we do not have an answer, even in special cases or for elementary examples, indicates there is something important about the structure of simple Leavitt path algebras that we do not yet understand. In addition, Open Question 1 is not only an impediment for classification of simple Leavitt path algebras, but until we have a solution it essentially impossible to classify nonsimple Leavitt path algebras of finite graphs—to do so, we would most likely need to deal with the simple ideals and quotients, which are as of yet unmanageable. Consequently, a solution to Open Question 1 is of paramount importance for the classification program for Leavitt path algebras.

Open Question 1 is compelling to the mathematical community not only for its applications to classification of algebras, but also because whatever the answer turns out to be, it will have consequences for the subjects of Algebra, Functional Analysis, and Dynamics. As the author sees it, there are three possible answers to Open Question 1: "Yes", "No", and "Sometimes".

If the answer is "Yes", then this would provide further compelling evidence for the existence of some sort of "Rosetta Stone" allowing for the translation of results between graph C^*-algebras and Leavitt path algebras. Identifying the reason for these similarities could lead to a deeper understanding of the relationships between C^*-algebras and algebras. It could even serve as a call to action for more collaboration between algebraists and analysts. Perhaps we can find conditions under which dense $*$-subalgebras of C^*-algebras have structural properties similar to their ambient C^*-algebras. Perhaps one can find larger classes of algebras for which analogues of C^*-algebra results can be proven. Or perhaps (if we dream big) a version of the Kirchberg-Phillips classification theorem could be proved for a large class of purely infinite simple algebras.

If the answer is "No", meaning the sign of the determinant is an invariant of Morita equivalence, then the pair $(K_0^{\text{alg}}(L_K(E)), \operatorname{sgn}(\det(1 - A_F^t)))$ would be a complete Morita equivalence invariant for simple Leavitt path algebras of finite graphs. This would imply that for finite strongly connected graphs, the Morita equivalence class of the Leavitt path algebra coincides exactly with the flow equivalence class of the graph's shift space (cf. Theorem 4.1). Consequently, we would have that the Leavitt path algebras and shift spaces are intimately related, suggesting that there is some deeper, not yet understood connection between the algebras and the flow dynamics.

If the answer is "Sometimes", meaning that for certain graphs changing the sign of the determinant (or performing a Cuntz splice) changes the Morita equivalence class of the Leavitt path algebra but for other graphs it does not, then we will need to identify exactly which graphs are affected. This would be the most surprising (and hence for a mathematician the most interesting!) outcome to this question. If indeed the answer does turn out to be "Sometimes", this outcome will likely motivate the creation and development of new tools and require the collaboration of algebraists, analysts, and dynamicists to investigate the phenomena that occur.

6 Classification of Leavitt Path Algebras of Infinite Graphs

As we saw in the previous section, the sign of the determinant condition (and unknown effect of the Cuntz splice) creates an impediment to classifying simple Leavitt path algebras of finite graphs

However, if we continue to look to graph C^*-algebras for inspiration, we see that Sørensen's classification of unital C^*-algebras of infinite graphs avoided the sign of the determinant and no Cuntz splice move was needed. One could therefore hope for a similar classification, using Sørensen's techniques, for unital Leavitt path algebras of infinite graphs. (Such graphs have a finite number of vertices and an infinite number of edges.) This collection of graphs, while avoiding the Cuntz splice, introduces a new problem: What is our candidate for the complete Morita equivalence invariant? For C^*-algebras of finite graphs we used the K_0-group, and when we considered Leavitt path algebras of finite graphs, we were in the fortunate situation that for any graph E we have $K_0^{\mathrm{alg}}(L_K(E)) \cong K_0(C^*(E))$. However, for graphs with infinitely many edges Sørensen now had to include the K_1-group of the C^*-algebra. In general, for a graph E one has that $K_1^{\mathrm{alg}}(L_K(E))$ and $K_1(C^*(E))$ are not equal. In addition, due to Bott periodicity, a C^*-algebras really only has only two K-groups: the K_0-group and then K_1-group. This means that by using K_0 and K_1, Sørensen was including *all* the K-groups of the graph C^*-algebra in the invariant. For Leavitt path algebras there is no periodicity and the algebraic K-groups $K_n^{\mathrm{alg}}(L_K(E))$ may all be distinct. Furthermore, for $n \geq 1$ one has that $K_n^{\mathrm{alg}}(L_K(E))$ depends on the underlying field of the Leavitt path algebra. This raises the question as to which of the algebraic K-groups $K_n^{\mathrm{alg}}(L_K(E))$ should be included in the invariant. K_0^{alg} and K_1^{alg} only? Some finite number of algebraic K-groups? All algebraic K-groups? Should the number of K-groups included depend on the field?

Inspired by the techniques of Sørensen in [19], Ruiz and the author looked for an invariant that would provide the moves needed between the graphs without worrying about whether this invariant involved the algebraic K-groups. It was found that a complete Morita equivalence invariant is provided by the pair $(K_0^{\mathrm{alg}}(L_K(E)), |E^0_{\mathrm{sing}}|)$. Here $|E^0_{\mathrm{sing}}|$ is the cardinality of the set of singular vertices E^0_{sing}. (Recall that a vertex is singular if it either emits no edges or an infinite number of edges.) Ruiz and the author proved the following in [18].

Theorem 6.1 (Ruiz and Tomforde) *Let K be a field, and let E and F be graphs with a finite number of vertices and an infinite number of edges with the property that $L_K(E)$ and $L_K(F)$ are simple. Then the following are equivalent:*

(1) $L_K(E)$ is Morita equivalent to $L_K(F)$.
(2) $K_0^{\mathrm{alg}}(L_K(E)) \cong K_0^{\mathrm{alg}}(L_K(F))$ and $|E^0_{\mathrm{sing}}| = |F^0_{\mathrm{sing}}|$.
(3) The graph E may turned into the graph F by finitely many applications of Moves (S), (O), (I), (R), their inverses.

While this result shows that $(K_0^{\mathrm{alg}}(L_K(E)), |E^0_{\mathrm{sing}}|)$ is a complete Morita equivalence invariant for unital simple Leavitt path algebras of infinite graphs, this

invariant is unsatisfying because it depends on the choice of the graph used to represent the Leavitt path algebra, rather than only on intrinsic properties of the algebra itself. We prefer to have an invariant based solely on algebraic properties, and if our goal is to lay groundwork for classification of larger classes of algebras, we hope that we can obtain an invariant described entirely in terms of K-theory.

Thus we ask whether or not some collection of the algebraic K-groups can provide a complete Morita equivalence invariant for unital simple Leavitt path algebras of infinite graphs—or, equivalently, whether $|E^0_{\mathrm{sing}}|$ can be determined from some collection of the algebraic K-groups.

In [18] Ruiz and the author showed that in certain situations the answer is "Yes", and surprisingly the answer depends on the underlying field. (This is explained by the fact that the higher algebraic K-groups of the Leavitt path algebra depend significantly on the underlying field.) To describe the manageable fields, we need to introduce a bit of terminology.

Definition 6.2 if G is an abelian group, we say *G has no free quotients* if no nonzero quotient of G is a free abelian group. If K is a field, we say *K has no free quotients* if the multiplicative abelian group $K^\times := K \setminus \{0\}$ has no free quotients.

It is shown in [18, Proposition 6.10] that the following are all examples of fields with no free quotients.

- All fields K such that $K^\times$ is a torsion group.
- All fields K such that $K^\times$ is weakly divisible.
- All algebraically closed fields.
- All fields that are perfect with characteristic $p > 0$.
- All finite fields.
- The field $\mathbb{C}$ of complex numbers.
- The field $\mathbb{R}$ of real numbers.

The field $\mathbb{Q}$ is not a field with no free quotients, because $\mathbb{Q}^\times \cong \mathbb{Z}_2 \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \ldots$.

In [18] Ruiz and the author showed that when the underlying field has no free quotients, the pair $(K_0^{\mathrm{alg}}(L_K(E)), K_1^{\mathrm{alg}}(L_K(E)))$ is a complete Morita equivalence invariant. We emphasize that this includes the case when the underlying field is the complex numbers.

Theorem 6.3 (Ruiz and Tomforde) *Let K be a field with no free quotients (see Definition 6.2), and let E and F be graphs with a finite number of vertices and an infinite number of edges, and with the property that $L_K(E)$ and $L_K(F)$ are simple. Then the following are equivalent:*

(1) $L_K(E)$ is Morita equivalent to $L_K(F)$.
(2) $K_0^{\mathrm{alg}}(L_K(E)) \cong K_0^{\mathrm{alg}}(L_K(F))$ and $K_1^{\mathrm{alg}}(L_K(E)) \cong K_1^{\mathrm{alg}}(L_K(F))$.
(3) The graph E may turned into the graph F by finitely many applications of Moves (S), (O), (I), (R), their inverses.

Moreover, in [18, §11] Ruiz and the author produce an example of graphs E and F with finitely many vertices, infinitely many edges, and having the

following properties: $L_{\mathbb{Q}}(E)$ and $L_{\mathbb{Q}}(F)$ are simple, $K_0^{\mathrm{alg}}(L_{\mathbb{Q}}(E)) \cong K_0^{\mathrm{alg}}(L_{\mathbb{Q}}(F))$, $K_1^{\mathrm{alg}}(L_{\mathbb{Q}}(E)) \cong K_1^{\mathrm{alg}}(L_{\mathbb{Q}}(F))$, and $K_2^{\mathrm{alg}}(L_{\mathbb{Q}}(E)) \ncong K_2^{\mathrm{alg}}(L_{\mathbb{Q}}(F))$. Thus the K_0^{alg}-groups and K_1^{alg}-groups of $L_{\mathbb{Q}}(E)$ and $L_{\mathbb{Q}}(F)$ are isomorphic, but $L_{\mathbb{Q}}(E)$ and $L_{\mathbb{Q}}(F)$ are not Morita equivalent. Hence the pair of the K_0^{alg}-group and K_1^{alg}-group can fail to be a complete Morita equivalence invariant when the underlying field is not a field with no free quotients.

This raises the question of whether higher algebraic K-groups can be included to produce a complete Morita equivalence invariant for other fields. This was answered affirmatively for number fields in [10]. (Recall that a number field is a field that is a finite extension of $\mathbb{Q}$.) The following was proven in [10].

Theorem 6.4 (Gabe, Ruiz, Tomforde, and Whalen) *Let K be a number field, and let E and F be graphs with a finite number of vertices and an infinite number of edges, and with the property that $L_K(E)$ and $L_K(F)$ are simple. Then the following are equivalent:*

(1) $L_K(E)$ is Morita equivalent to $L_K(F)$.
(2) $K_0^{\mathrm{alg}}(L_K(E)) \cong K_0^{\mathrm{alg}}(L_K(F))$ and $K_6^{\mathrm{alg}}(L_K(E)) \cong K_6^{\mathrm{alg}}(L_K(F))$.
(3) The graph E may turned into the graph F by finitely many applications of Moves (S), (O), (I), (R), their inverses.

This shows that the pair $(K_0^{\mathrm{alg}}(L_K(E)), K_6^{\mathrm{alg}}(L_K(E)))$ is a complete Morita equivalence invariant for these Leavitt path algebras when the field is a number field. In addition, since $\mathbb{Q}$ is a number field, this result covers the example produced by Ruiz and the author in [18, §11].

The situation for other fields is unclear, and this leads us to an important open question.

Open Question 2 *Let K be a field, and let E and F be graphs with a finite number of vertices and an infinite number of edges, and with the property that $L_K(E)$ and $L_K(F)$ are simple. If $K_n^{\mathrm{alg}}(L_K(E)) \cong K_n^{\mathrm{alg}}(L_K(F))$ for all $n \in \mathbb{N} \cup \{0\}$, then is it the case that $L_K(E)$ is Morita equivalent to $L_K(F)$?*

In light of Theorem 6.1, Open Question 2 is equivalent to asking the following: "If K is a field and E is a graph with a finite number of vertices and an infinite number of edges, and for which $L_K(E)$ is simple, then is it possible to determine $|E^0_{\mathrm{sing}}|$ from the set of algebraic K-groups $\{K_n^{\mathrm{alg}}(L_K(E)) : n = 0, 1, 2 \ldots\}$?"

Theorems 6.3 and 6.4 show that Open Question 2 has an affirmative answer when the field either has no free quotients or is a number field. This means that for simple Leavitt path algebras over these fields the only missing part of a classification is to answer Open Question 1 and determine whether the sign of the determinant is a Morita equivalence invariant in the case of finite graphs.

To obtain a classification of all simple Leavitt path algebras by algebraic K-theory, a positive answer to Open Question 2 is necessary. If the answer to Question 2 is negative in general, then a general classification in terms of algebraic K-theory will not be possible and we will need to restrict our attention to Leavitt

path algebras over particular fields. Fortunately, for many fields of interest (e.g., $\mathbb{C}$, $\mathbb{R}$, finite fields, $\mathbb{Q}$, number fields) we already know that Open Question 2 has a positive answer.

Acknowledgements The author thanks the organizers of the 2015 Abel Symposium, Christian Skau (Norwegian University of Science and Technology), Toke M. Carlsen (University of the Faroe Islands), Nadia Larsen (University of Oslo) and Sergey Neshveyev (University of Oslo) for their hospitality and the opportunity to attend. This work, including the author's travel to the Abel Symposium, was supported by a grant from the Simons Foundation (#210035 to Mark Tomforde).

References

1. G. Abrams and G. Aranda Pino, *The Leavitt path algebra of a graph*, J. Algebra **293** (2005), no. 2, 319–334.
2. G. Abrams, A. Louly, E. Pardo, and C. Smith, *Flow invariants in the classification of Leavitt path algebras*, J. Algebra **333** (2011), 202–231.
3. J. Cuntz, *K-theory for certain C^*-algebras*, Ann. Math. **113** (1981), 181–197.
4. J. Cuntz and W. Krieger, *A class of C^*-algebras and topological Markov chains*, Invent. Math. **56** (1980), 251–268.
5. D. Drinen, *Viewing AF-algebras as graph algebras*, Proc. Amer. Math. Soc. **128** (2000), 1991–2000.
6. S. Eilers, G. Restorff, and E. Ruiz, *Classification of graph C^*-algebras with no more than four primitive ideals*, Operator algebra and dynamics, 89–129, Springer Proc. Math. Stat., 58, Springer, Heidelberg, 2013.
7. S. Eilers, G. Restorff, E. Ruiz, and A. P. W. Sørensen, *Geometric classification of unital graph C^*-algebras of real rank zero*, preprint, 2015. `arXiv:1505.06773 [math.OA]`
8. S. Eilers and M. Tomforde, *On the classification of nonsimple graph C^*-algebras*, Math. Ann. **346** (2010), 393–418.
9. G.A. Elliott, and A. Toms, *Regularity properties in the classification program for separable amenable C^*-algebras*, Bull. Amer. Math. Soc. (N.S.) **45** (2008), no. 2, 229–245.
10. J. Gabe, E. Ruiz, M. Tomforde, and T. Whalen, *K-theory for Leavitt path algebras: computation and classification*, J. Algebra **433** (2015), 35–72.
11. R. Johansen and A. P. W. Sørensen, *The Cuntz splice does not preserve $*$-isomorphism of Leavitt path algebras over $\mathbb{Z}$*, preprint, 2015. `arXiv:1507.01247 [math.RA]`
12. D. Lind and B. Marcus, An Introduction to Symbolic Dynamics and Coding, Cambridge University Press, Cambridge, 1995.
13. R. Meyer and R. Nest, Ryszard, *C^*-algebras over topological spaces: filtrated K-theory*, Canad. J. Math. **64** (2012), no. 2, 368–408.
14. N. C. Phillips, *A classification theorem for nuclear purely infinite simple C^*-algebras*, Doc. Math. **5** (2000), 49–114.
15. M. Rørdam, *Classification of Cuntz-Krieger algebras*, *K*-theory **9** (1995), 31–58.
16. M. Rørdam, Classification of Nuclear, Simple C^*-algebras, Encyclopaedia of Mathematical Sciences, vol. 126, Springer, Berlin, 2001.
17. M. Rørdam, *Structure and classification of C^*-algebras*, International Congress of Mathematicians. Vol. II, 1581–1598, Eur. Math. Soc., Zürich, 2006.
18. E. Ruiz, and M. Tomforde, *Classification of unital simple Leavitt path algebras of infinite graphs*, J. Algebra **384** (2013), 45–83.

19. A. P. W. Sørensen, *Geometric classification of simple graph algebras*, Ergodic Theory Dynam. Systems **33** (2013), no. 4, 1199–1220.
20. W. Szymański, *The range of K-invariants for C^*-algebras of infinite graphs*, Indiana Univ. Math. J. **51** (2002), 239–249.
21. A. Tikuisis, S. White, and W. Winter, *Quasidiagonality of nuclear C^*-algebras*, preprint (2015).
22. M. Tomforde, *Leavitt path algebras with coefficients in a commutative ring*, J. Pure Appl. Algebra **215** (2011), 471–484.

QDQ vs. UCT

Wilhelm Winter

Abstract This is a survey of recent progress in the structure and classification theory of nuclear C*-algebras. In particular, I outline how the Universal Coefficient Theorem ensures a positive answer to the quasidiagonality question in the presence of faithful traces. This has strong consequences for the regularity conjecture and the classification problem for separable, simple, nuclear C*-algebras. Moreover, it entails a positive solution to Rosenberg's conjecture on quasidiagonality of reduced C*-algebras of discrete amenable groups. This note is largely based on a joint paper with Aaron Tikuisis and Stuart White.

1 Introduction

Quasidiagonality was defined by Halmos as an external finite dimensional approximation property for sets of operators on a Hilbert space [25]. Voiculescu studied the notion as a property of (represented) C*-algebras; cf. [56, 57].

For nuclear C*-algebras quasidiagonality can be expressed as an embedding property via the Choi–Effros lifting theorem [7]. Another connection to amenability was disclosed by Hadwin in [24] and—in an appendix to the same paper—by Rosenberg who observed that discrete group C*-algebras can only be quasidiagonal for amenable groups. The converse statement became known as Rosenberg's conjecture.

Elliott's programme to classify nuclear C*-algebras by K-theory data first featured quasidiagonality in the proof of Kirchberg's embedding theorem [28] (where it entered through Voiculescu's homotopy invariance theorem [56]), which then led up to the celebrated Kirchberg–Phillips classification of simple, purely infinite, nuclear C*-algebras; see [30, 44].

In the stably finite case, the relevance of quasidiagonality was first marked by Popa's local quantisation of [42], which then inspired Lin to define TAF algebras,

W. Winter (✉)
Mathematisches Institut der Westfälischen Wilhelms-Universität Münster, Einsteinstraße 62, 48149 Münster, Germany
e-mail: wwinter@uni-muenster.de

T.M. Carlsen et al. (eds.), *Operator Algebras and Applications*, Abel Symposia 12,
DOI 10.1007/978-3-319-39286-8_15

thus leading up to the whole TAF/TAI classification machinery (which recently culminated in the spectacular [22] and [17] by Elliott, Gong, Lin, and Niu).

The quasidiagonality question (QDQ) of Blackadar and Kirchberg asks whether all separable, stably finite, nuclear C^*-algebras are quasidiagonal; see [4]. In the 2000s, the interest in the interplay between quasidiagonality and nuclearity was renewed in Brown's [6] (which takes a measure theoretic or tracial point of view) and in [32] (which establishes a further link to topological approximation properties via the decomposition rank, a noncommutative covering dimension). The latter notion was generalised to nuclear dimension in [63], and it became an intriguing problem to characterise the difference between decomposition rank and nuclear dimension.

A massive breakthrough was achieved by Matui and Sato in [36]: In the simple, monotracial and $\mathcal{Z}$-absorbing case (where $\mathcal{Z}$ denotes the Jiang–Su algebra from [27]; see also [47]) quasidiagonality implies finite decomposition rank; in the absence of quasidiagonality, by [49] one always has finite nuclear dimension. For larger trace spaces, one has to consider Brown's more refined notion of quasidiagonality of traces; cf. [5].

The latter condition also appears in [17] which amounts to the classification of all separable, unital, simple C^*-algebras with finite nuclear dimension, which satisfy the Universal Coefficient Theorem from [48] (UCT for short; satisfying it is equivalent to being KK-equivalent to a commutative C^*-algebra) and for which all traces are quasidiagonal. Although of stunning generality, the last two hypotheses in this result do remain mysterious. The UCT holds for C^*-algebras of amenable groupoids by [55] and is also ensured by a bootstrap type condition; therefore in concrete situations of interest it can usually be confirmed (see, for example, [15]). To date it is unknown whether *all* separable nuclear C^*-algebras satisfy the UCT. Quasidiagonality—due to its poor permanence properties—is usually also hard to verify even in very concrete and geometric setups; see for example [40] or [34].

Remarkably, neither of these two problems occur in the von Neumann algebra context of Connes' celebrated classification of injective factors (which remains an inspiration for the classification of nuclear C^*-algebras—at a philosophical, but also at a technical level, since the various available proofs in [9, 23, 41] have revealed important insights for C^*-algebraists).

The situation on the C^*-algebra side changed considerably with [51], in which the two problems were linked to the effect that the UCT indeed ensures quasidiagonality under suitable circumstances, with the crucial extra tool being stable uniqueness theorems as introduced in [11, 33]. The result verifies Rosenberg's conjecture and has strong consequences for the structure and classification of simple nuclear C^*-algebras, including the regularity conjecture of Toms and myself (see [18, 59]). In particular, the classification of separable, unital, simple C^*-algebras which satisfy the UCT and have finite nuclear dimension is now complete. Moreover, the nuclear dimension hypothesis may be replaced by $\mathcal{Z}$-absorption or by strict comparison of positive elements, at least under additional conditions on the space of tracial states.

In this note I give a survey of the result, its proof and its corollaries. I try to motivate it and put it into a larger context—especially in view of the remaining open problems in the area—by specialising to the case of strongly self-absorbing C*-algebras.

In Sect. 2 I recall the concept of and some elementary facts about quasidiagonality. In Sect. 3 I state the stable uniqueness theorems and outline the role of the UCT for the controlled version. Section 4 looks at strongly self-absorbing C*-algebras and approaches various versions of the quasidiagonality question from this point of view. Section 5 contains the main result and its consequences for structure and classification of nuclear C*-algebras and for Rosenberg's conjecture. The proof of the main theorem is outlined in Sect. 6. Finally, Sect. 7 collects a number of open problems, mostly on quasidiagonality and on the UCT.

Most of the contents of the paper are based on [51], except for those of Sections 4 and 7 which have not previously been published in this form. I would like to thank Ilijas Farah, David Kerr, Aaron Tikuisis, Stuart White and the referee for some very helpful comments on an earlier version.

This note was written during a research stay at the Mittag-Leffler Institute to which I am indebted for their hospitality. I am grateful to the Deutsche Forschungsgemeinschaft for their support through the collaborative research centre SFB 878. And I would like to thank the organisers of the 2015 Abel Symposium for their kind invitation to this marvellous conference.

2 Quasidiagonality

2.1 Quasidiagonality was originally defined by Halmos for sets of operators on a Hilbert space; an abstract C*-algebra is called quasidiagonal if it has a faithful representation the image of which is quasidiagonal in this sense. For our purposes, it will be useful to take Voiculescu's characterisation from [56] as a definition:

Definition A C*-algebra A is quasidiagonal if, for every finite subset $\mathcal{F}$ of A and $\varepsilon > 0$, there exist $k \in \mathbb{N}$ and a completely positive contractive (c.p.c.) map $\psi : A \to M_k$ such that

$$\|\psi(ab) - \psi(a)\psi(b)\|, \ \|a\| - \|\psi(a)\| < \varepsilon$$

for all $a, b \in \mathcal{F}$.

2.2 With the aid of the Choi–Effros lifting theorem it is not hard to see that for separable nuclear C*-algebras quasidiagonality can be rephrased in terms of embeddings into $\mathcal{Q}_\omega$, where $\mathcal{Q}$ is the universal UHF algebra, $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ is a free ultrafilter, and $\mathcal{Q}_\omega = \prod_{\mathbb{N}} \mathcal{Q}/\{(x_n)_{\mathbb{N}} \mid \lim_{n\to\omega} x_n = 0\}$.

Proposition *A separable nuclear* C*-*algebra A is quasidiagonal if and only if there is an embedding* $A \hookrightarrow \mathcal{Q}_\omega$.

2.3 *Remark* Note that in this characterisation unitality is not an issue: A as above is quasidiagonal if and only if the smallest unitisation $A^{\sim}$ is quasidiagonal if and only if there is a unital embedding $A^{\sim} \hookrightarrow \mathcal{Q}_\omega$. (This last statement uses the fact that $\mathcal{Q}$ is self-similar in the sense that every unital hereditary C*-subalgebra of $\mathcal{Q}$ is isomorphic to $\mathcal{Q}$.)

2.4 Brown considered in [6] the refined notion of *quasidiagonal traces*; this is based on Voiculescu's observation [57] that a unital, separable, quasidiagonal C*-algebra always has at least one tracial state which is picked up by a sequence of (unital) quasidiagonal approximations as in Definition 2.1.

Definition A tracial state $\tau_A \in \mathrm{T}(A)$ on a C*-algebra A is quasidiagonal if, for every finite subset $\mathcal{F}$ of A and $\varepsilon > 0$, there exist $k \in \mathbb{N}$ and a c.p.c. map $\psi : A \to M_k$ such that

$$\|\psi(ab) - \psi(a)\psi(b)\|, \ |\tau_A(a) - \tau_{M_k} \circ \psi(a)| < \varepsilon$$

for all $a, b \in \mathcal{F}$.

2.5 Similar to Proposition 2.2, one can characterise quasidiagonality of traces in terms of maps to $\mathcal{Q}_\omega$.

Proposition *Let A be a separable nuclear* C*-*algebra. A tracial state* $\tau_A \in \mathrm{T}(A)$ *is quasidiagonal if and only if there is a* *-*homomorphism* $\pi : A \to \mathcal{Q}_\omega$ *such that* $\tau_A = \tau_{\mathcal{Q}_\omega} \circ \pi$, *where* $\tau_{\mathcal{Q}_\omega}$ *is the unique tracial state on* $\mathcal{Q}_\omega$ *given by* $\tau_{\mathcal{Q}_\omega}([(x_n)_{\mathbb{N}}]) = \lim_{n\to\omega} \tau_{\mathcal{Q}}(x_n)$.

3 Stable Uniqueness and the Universal Coefficient Theorem

3.1 It is a common problem in classification to decide when two morphisms are (approximately unitarily) equivalent, provided there is no obvious K-theoretic obstruction. Stable uniqueness theorems provide partial solutions by establishing 'local almost' unitary equivalences after adding the same map on both sides with sufficiently large multiplicity. (Very roughly speaking, this could perhaps be interpreted as a fine-tuned version of Voiculescu's theorem [58].) The problem has been studied extensively by Lin (see [33], for example); for our purposes a version of Dadarlat and Eilers—which we state here in a simplified form for brevity—will be particularly useful.

Theorem [11, Theorem 4.5] *Let* C, B *be unital* C*-*algebras with* C *separable and nuclear. Let* $\iota : C \to B$ *be a unital* *-*homomorphism which is totally full, i.e., for every nonzero* $c \in C$ *the element* $\iota(c)$ *generates* B *as an ideal. Let* $\varphi, \psi : C \to B$ *be unital* *-*homomorphisms with the same induced* KK-*class.*

Then, for every finite subset $\mathcal{G} \subset C$ and $\delta > 0$ there are $n \in \mathbb{N}$ and a unitary $u \in M_{n+1}(B)$ such that

$$\|u(\varphi(c) \oplus \iota^{\oplus n}(c))u^* - (\psi(c) \oplus \iota^{\oplus n}(c))\| < \delta, \quad c \in \mathcal{G}.$$

3.2 *Remark* For us it will be important that the domain algebra C in the theorem above need not be simple. In fact, we will apply it to the unitisation of the suspension of the algebra we are actually interested in. For the target we mostly care about the algebra $\mathcal{Q}_\omega$; for technical reasons we also consider algebras like $\prod_{\mathbb{N}} \mathcal{Q}_\omega$.

3.3 For the proof of our main result, we need a refined version of Theorem 3.1, since for our method it is vital that n can be chosen independent of the specific maps φ, ψ and especially ι (it may of course still depend on C, $\mathcal{G}$ and δ). Such a result was already provided in [11, Theorem 4.12], at least for simple domains C. However, the proof carries over to the nonsimple situation as long as one retains some control over the fullness of ι. This is done in [51] in terms of a control function $\Delta : C_+^1 \setminus \{0\} \to \mathbb{N}$ (where C_+^1 is the unit ball of the positive elements of C and $\Delta(c)$ is the smallest $k \in \mathbb{N}$ such that there are contractions $x_1, \ldots, x_k \in B$ with $1_B = \sum_{i=1}^k x_i^* \iota(c) x_i$). For a given *-homomorphism $\iota : C \to B$ one can define an (a priori possibly infinite) function Δ in terms of (inverses of) tracial values of $\iota(c)$ for $c \in C_+^1 \setminus \{0\}$; provided B has strict comparison of positive elements and the control function remains finite, the map ι will then be Δ-full—see [51] for details.

The proof of [11, Theorem 4.12] works by assuming the statement is false and then producing a sequence of pairs of counterexamples (i.e., pairs of maps as in Theorem 3.1, but with no uniform bound on n) for contradiction. The crux is it to assemble the two resulting sequences of maps into two single maps (with larger target algebras) and at the same time keeping control over their KK-classes. In our situation, the individual maps will be zero-homotopic, and the problem is it to decide when maps of the form

$$\mathrm{KK}\big(C, \textstyle\prod_{\mathbb{N}} B_n\big) \to \textstyle\prod_{\mathbb{N}} \mathrm{KK}(C, B_n) \tag{3.1}$$

are injective. The issue is that a sequence of homotopies doesn't necessarily give rise to a (continuous) homotopy of sequences, since the parameter speeds might increase quickly. It is for this purpose—and for this purpose only—that the Universal Coefficient Theorem enters the stage.

3.4 Definition [48, Theorem 1.17], [3] A separable C*-algebra A is said to satisfy the Universal Coefficient Theorem (UCT) if the sequence

$$0 \to \mathrm{Ext}(\mathrm{K}_*(A), \mathrm{K}_{*+1}(B)) \to \mathrm{KK}(A, B) \to \mathrm{Hom}(\mathrm{K}_*(A), \mathrm{K}_*(B)) \to 0$$

is exact for every σ-unital C*-algebra B.

Both maps in the sequence above can be made explicit, using the natural identification $\mathrm{KK}(A, B) \cong \mathrm{Ext}^{-1}(A, \mathcal{C}_0(\mathbb{R}) \otimes B \otimes \mathcal{K})$: Given such an extension,

the right hand map collects the boundary maps of the associated six-term exact sequence in K-theory. For trivial boundary maps, the six-term exact sequence splits into two extensions of abelian groups; the UCT requires the left-hand map to be the inverse of this assignment.

3.5 It is known that the separable C^*-algebras satisfying the UCT are precisely the ones which are KK-equivalent to abelian C^*-algebras; see [48], [50, Proposition 5.3] (or [3, Theorem 23.10.5]). The closure properties of the class of *nuclear* UCT C^*-algebras are so strong that to date nobody has managed to find a nuclear example outside it. The UCT problem reads as follows.

Question Does every separable nuclear C^*-algebra satisfy the UCT?

3.6 In our situation, the UCT provides just enough information to make the map of (3.1) injective; see [11] and [51] for details. Let us state the resulting 'controlled' stable uniqueness theorem, again for brevity in a simplified version. (See [51] for the complete statement; this involves K-theory with coefficients, which here boils down to just ordinary K-theory by $\mathcal{Q}$-stability.)

Theorem *Let C be a separable, unital, nuclear C^*-algebra satisfying the UCT. Let $\Delta : C^1_+ \setminus \{0\} \to \mathbb{N}$ be a control function, let $\mathcal{G} \subset C$ be a finite subset and let $\delta > 0$. Then there exists $n \in \mathbb{N}$ such that for any unital Δ-full *-homomorphism $\iota : C \to B$ (where B is of the form $\mathcal{Q}$, $\mathcal{Q}_\omega$ or $\prod_{\mathbb{N}} \mathcal{Q}_\omega$), and any unital *-homomorphisms $\varphi, \psi : C \to B$ with $\mathrm{K}_*(\varphi) = \mathrm{K}_*(\psi)$, there is a unitary $u \in M_{n+1}(B)$ such that*

$$\|u(\varphi(c) \oplus \iota^{\oplus n}(c))u^* - (\psi(c) \oplus \iota^{\oplus n}(c))\| < \delta, \quad c \in \mathcal{G}.$$

4 QDQ: A Strongly Self-Absorbing Point of View

4.1 The quasidiagonality question, also known as Blackadar–Kirchberg problem, was posed in [4]:

QDQ *Is every separable, stably finite, nuclear C^*-algebra quasidiagonal?*

Note that by Remark 2.3, **QDQ** is equivalent to $\mathbf{QDQ}_1$, the respective question for unital C^*-algebras.

Likewise, one could add simplicity or both unitality and simplicity to the hypotheses; we denote the resulting questions by $\mathbf{QDQ}_{\mathrm{simple}}$ and $\mathbf{QDQ}_{\mathrm{simple},1}$. At this point I am not aware of any (but the obvious) formal implications between **QDQ**, $\mathbf{QDQ}_{\mathrm{simple}}$ and $\mathbf{QDQ}_{\mathrm{simple},1}$, although it doesn't seem unlikely that the general case can be reduced to the simple situation (for example via some Bernoulli type crossed product construction as in [39]?).

4.2 Although the quasidiagonality question has been around for a long time, until recently its role for the structure and classification theory of nuclear C^*-algebras has

remained somewhat obscure, and its nature and complexity is still hard to gauge. Maybe most conspicuously, it seems to be a very finite problem, in stark contrast to other structural phenomena, which often exhibit a dichotomy between finite and infinite situations, such as the interplay between Lin's TAF classification and Kirchberg–Phillips classification, the roles of the Jiang–Su algebra $\mathcal{Z}$ and the Cuntz algebra $\mathcal{O}_\infty$, and Toms' and Rørdam's topologically high-dimensional examples [45, 52].

4.3 In order to shed some light on the quality of the problem, let us be more restrictive and formulate the quasidiagonality question for strongly self-absorbing C*-algebras. These have been abstractly defined in [53], but the concept goes back to Effros–Rosenberg [16] in the C*-algebra setting, which in turn was inspired by the von Neumann algebra context where it was crucial for McDuff's [38] and for Connes' celebrated classification of injective factors [9].

4.4 Definition A separable unital C*-algebra $\mathcal{D} \neq \mathbb{C}$ is strongly self-absorbing, if there is an isomorphism $\varphi : \mathcal{D} \to \mathcal{D} \otimes \mathcal{D}$ which is approximately unitarily equivalent to the first factor embedding, i.e., there is a sequence of unitaries $(u_n)_\mathbb{N}$ in $\mathcal{D} \otimes \mathcal{D}$ such that $u_n(d \otimes 1_\mathcal{D})u_n^* \stackrel{n\to\infty}{\longrightarrow} \varphi(d)$ for all $d \in \mathcal{D}$.

4.5 It was shown in [16] that strongly self-absorbing C*-algebras are always simple and nuclear; from results of Kirchberg it follows that they are either purely infinite or stably finite and in this case there is a unique tracial state (cf. [53]). By [60] they are always $\mathcal{Z}$-stable, i.e. they absorb $\mathcal{Z}$ tensorially.

4.6 For strongly self-absorbing C*-algebras tensorial absorption can be characterised in terms of (exact or approximate) unital embeddings: A separable unital C*-algebra A absorbs the strongly self-absorbing C*-algebra $\mathcal{D}$ precisely if $\mathcal{D}$ embeds unitally into the commutant of A inside its ultrapower, $A_\omega \cap A'$ (the criterion can also be phrased for nonunital A). The proof is based on an Elliott intertwining argument; see [44, Theorem 7.2.2] or [53, Theorem 2.3] (these are stated for sequence algebras instead of ultrapowers, but the proofs are essentially the same). For a strongly self-absorbing target $\mathcal{E}$, a separable subspace of the ultrapower $\mathcal{E}_\omega$ can be unitarily conjugated to a subspace of the relative commutant $\mathcal{E}_\omega \cap \mathcal{E}'$. As a consequence, two strongly self-absorbing C*-algebras $\mathcal{D}$ and $\mathcal{E}$ are isomorphic if and only if there are unital embeddings $\mathcal{D} \hookrightarrow \mathcal{E}$ and $\mathcal{E} \hookrightarrow \mathcal{D}$ if and only if there are unital embeddings $\mathcal{D} \hookrightarrow \mathcal{E}_\omega$ and $\mathcal{E} \hookrightarrow \mathcal{D}_\omega$; see [16] and [53]. Upon combining this with Proposition 2.2, we see that a (necessarily finite) strongly self-absorbing $\mathcal{D}$ is quasidiagonal if and only if $\mathcal{D} \otimes \mathcal{Q} \cong \mathcal{Q}$. Moreover, Kirchberg's embedding theorem yields that $\mathcal{O}_2$ absorbs any other strongly self-absorbing C*-algebra, $\mathcal{O}_2 \otimes \mathcal{D} \cong \mathcal{O}_2$. (This last statement holds in much greater generality.)

4.7 *Examples* The chart below contains all known strongly self-absorbing C*-algebras. Here, UHF$^\infty$ stands collectively for UHF algebras of infinite type. (The

universal UHF algebra $\mathcal{Q}$ is one of them; we list it separately to emphasise its role as a 'semifinal' object.) An arrow means 'embeds unitally into' or equivalently 'is tensorially absorbed by'.

$$\begin{array}{ccc}
 & & \mathcal{O}_2 \\
 & \nearrow & \uparrow \\
\mathcal{Q} & \to & \mathcal{Q} \otimes \mathcal{O}_\infty \\
\uparrow & & \uparrow \\
\mathrm{UHF}^\infty & \to & \mathrm{UHF}^\infty \otimes \mathcal{O}_\infty \\
\uparrow & & \uparrow \\
\mathcal{Z} & \to & \mathcal{O}_\infty
\end{array}$$

Arguably the most important question about strongly self-absorbing C*-algebras is whether or not the list above is complete. This makes direct contact with fundamental open problems such as the classification problem, the Toms–Winter conjecture, the UCT problem, or the quasidiagonality question. Even though being strongly self-absorbing is a very restrictive condition, at this point there is no evidence these questions will be substantially easier to answer when restricted to the strongly self-absorbing situation. On the other hand, such a restriction can often bare the problem of merely technical additional complications, and in this way sometimes disclose its true nature. Occasionally, a solution in the strongly self-absorbing case will then even give us a clue of how to deal with the general situation. This has happened for example in the run-up to [49] and to [51]; it is one of the reasons why I like to think of strongly self-absorbing C*-algebras as a microcosm within the world of all nuclear C*-algebras.

4.8 It is a crucial feature of the point of view above that questions on the existence or non-existence of examples with certain properties can be rephrased in terms of abstract characterisations of the known examples. For instance, the Jiang–Su algebra $\mathcal{Z}$ was characterised in [60] as the uniquely determined initial object in the category of all strongly self-absorbing C*-algebras. (An object in a category is initial, if there is a morphism to every other object. Very often this morphism is also required to be unique; in our situation, this will be the case when using approximate unitary equivalence classes instead of just unital *-homomorphisms.) At the opposite end, $\mathcal{O}_2$ is the unique final object (i.e., there is a morphism from every other object to $\mathcal{O}_2$; as above, this will be unique when using as morphisms approximate unitary equivalence classes of unital *-homomorphisms) by Kirchberg's embedding theorem. These are, as Kirchberg once put it, *sociological* characterisations, based

on interactions with peer objects. In [12], it was observed that $\mathcal{O}_2$ can also be characterised intrinsically—or *genetically*—as the unique strongly self-absorbing C^*-algebra with trivial K_0-group. Conspicuously, this characterisation of the *final* object does not require the UCT; in contrast, Kirchberg has shown that the UCT problem is in fact equivalent to the question whether a unital Kirchberg algebra with trivial K-theory is isomorphic to $\mathcal{O}_2$ [29], and Dadarlat has a parallel result featuring $\mathcal{Q}$ [10]. It is tempting to think of $\mathcal{Q}$ and $\mathcal{Q} \otimes \mathcal{O}_\infty$ in a similar way, as 'semifinal' objects: $\mathcal{Q}$ is final in the category of all *known* finite strongly self-absorbing C^*-algebras, and, more abstractly, also in the category of all quasidiagonal strongly self-absorbing C^*-algebras (cf. [16]). One can now turn the tables and interpret this fact as a characterisation of quasidiagonality for strongly self-absorbing C^*-algebras in terms of its final object. Similarly, $\mathcal{Q} \otimes \mathcal{O}_\infty$ is the final object in the category of all known strongly self-absorbing C^*-algebras which are not $\mathcal{O}_2$. Turning tables again one can look at the category of all strongly self-absorbing C^*-algebras which embed unitally into $\mathcal{Q} \otimes \mathcal{O}_\infty$ and interpret this as a notion of quasidiagonality which also makes sense in the infinite setting, at least in the strongly self-absorbing context.

4.9 The strongly self-absorbing version of the quasidiagonality question reads: *Is every finite strongly self-absorbing* C^**-algebra quasidiagonal?* In view of the preceding discussion, we obtain an equivalent formulation as follows:

QDQ$_{\text{finite s.s.a.}}$ *Is* $\mathcal{D} \otimes \mathcal{Q} \cong \mathcal{Q}$ *for every finite strongly self-absorbing* C^**-algebra* $\mathcal{D}$*?*

Note that this asks whether $\mathcal{Q}$ can be characterised abstractly as the final object in the category of finite strongly self-absorbing C^*-algebras. In the above one could specialise even a bit more and require the ordered K_0-group of $\mathcal{D}$ to be a subgroup of $\mathbb{Q}$ (with natural order).

4.10 Unlike the original quasidiagonality question, the version of 4.9 yields an obvious infinite counterpart by simply replacing $\mathcal{Q}$ with $\mathcal{Q} \otimes \mathcal{O}_\infty$ and 'finite' with the minimal necessary condition 'not isomorphic to $\mathcal{O}_2$':

QDQ$_{\text{infinite s.s.a.}}$ *Is* $\mathcal{D} \otimes \mathcal{Q} \otimes \mathcal{O}_\infty \cong \mathcal{Q} \otimes \mathcal{O}_\infty$ *for every strongly self-absorbing* C^**-algebra* $\mathcal{D}$ *not isomorphic to* $\mathcal{O}_2$*?*

Once again this asks for an abstract characterisation of $\mathcal{Q} \otimes \mathcal{O}_\infty$ as the final object in the category of all strongly self-absorbing C^*-algebras which are not $\mathcal{O}_2$ (or equivalently, which have nontrivial K-theory).

This infinite (or rather, general) version of the strongly self-absorbing quasidiagonality question runs completely parallel with its finite antagonist, and may be taken as first evidence that the original quasidiagonality question is just the finite incarnation of a much more general type of embedding problem.

4.11 We have now used a tool from classification—Elliott's intertwining argument—to rephrase the quasidiagonality question as an isomorphism problem, which makes sense both in a finite and an infinite context. Going only one step further, we

see that classification not only predicts, but in fact provides, a positive answer to $\mathbf{QDQ}_{\text{infinite s.s.a.}}$: The secret extra ingredient is to assume that $\mathcal{D}$ satisfies the UCT. Under this hypothesis, it was observed in [53] that the K-theory of $\mathcal{D}$ has to agree with that of one of the known strongly self-absorbing examples, and then it follows from Kirchberg–Phillips classification that $\mathcal{D}$ is indeed absorbed by $\mathcal{Q} \otimes \mathcal{O}_\infty$. We therefore have:

Theorem *If $\mathcal{D} \neq \mathcal{O}_2$ is a strongly self-absorbing* C^**-algebra which satisfies the UCT, then $\mathcal{D} \otimes \mathcal{Q} \otimes \mathcal{O}_\infty \cong \mathcal{Q} \otimes \mathcal{O}_\infty$.*

In other words, $\mathcal{Q} \otimes \mathcal{O}_\infty$ is the unique final object in the category of strongly self-absorbing C^**-algebras which have nontrivial* K*-theory and satisfy the UCT.*

With this observation at hand, I found it harder and harder to imagine $\mathbf{QDQ}_{\text{finite s.s.a.}}$ fails when also assuming the UCT. Now we know this perception was indeed correct (cf. 5.6 below), even in a generality going far beyond the strongly self-absorbing context (see 5.2). Here I took the strongly self-absorbing perspective mostly for a cleaner picture of a simpler situation—but with the benefit of hindsight, the theorem above provided just the necessary impetus to combine the quasidiagonality question with the UCT problem.

5 The Main Result: Structure and Classification

5.1 Theorem [51, Theorem A] *Let A be a separable nuclear* C^**-algebra which satisfies the UCT. Then every faithful trace on A is quasidiagonal.*

Short after the distribution of [51], Gabe observed in [19] that essentially the same argument works when weakening the nuclearity hypotheses to A being exact and the trace being amenable. Before outlining the proof of the theorem above let us list some consequences, mostly for the structure and classification of simple C^*-algebras, but also for Rosenberg's conjecture.

5.2 Corollary [51, Corollary B] *Every separable nuclear* C^**-algebra which satisfies the UCT and has a faithful trace is quasidiagonal. In particular, the quasidiagonality question has a positive answer for simple unital* C^**-algebras satisfying the UCT.*

5.3 In the appendix of [24], Rosenberg observed that for a discrete group G, if the reduced group C^*-algebra $\mathrm{C}^*_\mathrm{r}(G)$ is quasidiagonal then G is amenable. The converse was Rosenberg's conjecture, open since the 1980s. Our result in conjunction with [55] (which verifies the UCT assumption) confirms the conjecture (the canonical trace on $\mathrm{C}^*_\mathrm{r}(G)$ is well-known to be faithful). Together with Rosenberg's earlier result this yields a new characterisation of amenability for discrete groups. Note that at first glance our result seems to only cover countable discrete groups

(Theorem 5.1 deals with separable C*-algebras), but the general case follows since both quasidiagonality and amenability are *local* conditions.

Corollary [51, Corollary C] *For a discrete amenable group G, its reduced group* C**-algebra* $\mathrm{C}^*_{\mathrm{r}}(G)$ *is quasidiagonal.*

5.4 Elliott, Gong, Lin and Niu have very recently (see [17], which heavily uses [22]) obtained a spectacular classification result for unital simple nuclear C*-algebras—the crucial additional assumptions being finite decomposition rank and the UCT. They also show that finite decomposition rank may be weakened to finite nuclear dimension, provided all traces are quasidiagonal. Our Theorem 5.1 now shows that this last hypothesis is in fact redundant. This is important for applications, since finite nuclear dimension is notoriously easier to verify than finite decomposition rank, but it is also very satisfactory from a conceptual point of view, since for once it allows to state the purely infinite and the stably finite incarnations of classification in the same framework—and it also shows that quasidiagonality of traces precisely marks the dividing line between nuclear dimension and decomposition rank (at least in the simple UCT case), thus answering [63, Question 9.1] in this context.

Corollary [51, Corollary D] *The class of all separable, unital, simple, infinite dimensional* C**-algebras with finite nuclear dimension and which satisfy the UCT is classified by the Elliott invariant.*

5.5 It is worth highlighting the special case when there is at most one trace. For once, the statement becomes particularly clean then, partly because the classifying invariant reduces to just ordered K-theory in this situation, and moreover the proof only relies on work that has already been published (apart from [51]). The traceless case has been known for a long time—it is the by now classical Kirchberg–Phillips classification of purely infinite C*-algebras. The equivalence of conditions (i), (ii) and (iii) below in the tracial case is the culmination of [46, 61, 35, 49] and does not require the UCT; this only comes in to make the connection with (i').

Corollary [51, Corollaries E and 6.4] *The full Toms–Winter conjecture holds for* C**-algebras with at most one trace and which satisfy the UCT.*

That is, for a separable, unital, simple, infinite dimensional, nuclear C**-algebra A with at most one trace and with the UCT, the following are equivalent:*

(i) *A has finite nuclear dimension.*
(ii) *A is* $\mathcal{Z}$*-stable.*
(iii) *A has strict comparison of positive elements.*

If A is stably finite, then (i) *may be replaced by*

(i') *A has finite decomposition rank.*

Moreover, this class is classified up to $\mathcal{Z}$*-stability by ordered* K*-theory.*

5.6 Since strongly self-absorbing C*-algebras are $\mathcal{Z}$-stable by [60] and have at most one trace, we now know that the chart of 4.7 is indeed complete within the UCT class.

Corollary *The strongly self-absorbing* C*-*algebras satisfying the UCT are precisely the known ones.*

6 A Sketch of a Proof

6.1 In this outline of the proof of Theorem 5.1 I freely assume A to be unital, since one can easily reduce to this case. The very rough idea of the argument is it to produce two complementary cones over A and 'connect' them along the interval in order to construct an almost multiplicative map from $\mathcal{C}([0,1]) \otimes A$ to $M_2(\mathcal{Q}_\omega)$.

6.2 Let us begin by producing two cones over A in $\mathcal{Q}_\omega$ such that at least their scalar parts are compatible. In order to conjure up a single cone over A inside $\mathcal{Q}_\omega$ one might try to employ Voiculescu's theorem [56] on homotopy invariance of quasidiagonality, which will immediately yield an embedding of the cone over A into $\mathcal{Q}_\omega$. However, this method will typically give an embedding which is small in trace (not surprisingly, since Voiculescu's result works in complete generality, even when there are no traces around at all). For us this means that we won't be able to repeat the step in order to find the complementary second cone. Instead, we will need a more refined way of implementing quasidiagonality of cones. We will do this by carefully controlling tracial information for the embedding $\mathcal{C}_0((0,1], A) \hookrightarrow \mathcal{Q}_\omega$. Roughly speaking, we want the canonical trace on $\mathcal{Q}_\omega$ to be compatible with a prescribed trace on A and with Lebesgue measure on the interval. This was essentially laid out in [49] and refined in [51]; it heavily relies on Connes' [9] and also uses Kirchberg and Rørdam's [31].

Lemma *Let A be a separable, unital, nuclear* C*-*algebra and let $\tau_A \in \mathrm{T}(A)$ be a tracial state.*

(i) *There is a* *-*homomorphism*

$$\Psi : \mathcal{C}_0((0,1]) \otimes A \to \mathcal{Q}_\omega$$

such that

$$\tau_{\mathcal{Q}_\omega} \circ \Psi = \mathrm{ev}_1 \otimes \tau_A .$$

(ii) *There are* *-*homomorphisms*

$$\acute{\Phi} : \mathcal{C}_0((0,1]) \otimes A \to \mathcal{Q}_\omega ,$$

$$\grave{\Phi} : \mathcal{C}_0([0,1)) \otimes A \to \mathcal{Q}_\omega,$$

$$\Theta : \mathcal{C}([0,1]) \to \mathcal{Q}_\omega$$

which are compatible in the sense that

$$\acute{\Phi}|_{\mathcal{C}_0((0,1])\otimes 1_A} = \Theta|_{\mathcal{C}_0((0,1])} \quad \text{and} \quad \grave{\Phi}|_{\mathcal{C}_0([0,1))\otimes 1_A} = \Theta|_{\mathcal{C}_0([0,1))},$$

and such that

$$\tau_{\mathcal{Q}_\omega} \circ \acute{\Phi} = \tau_{\mathrm{Lebesgue}} \otimes \tau_A \quad \text{and} \quad \tau_{\mathcal{Q}_\omega} \circ \grave{\Phi} = \tau_{\mathrm{Lebesgue}} \otimes \tau_A.$$

We use τ_{Lebesgue} to denote the traces induced by Lebesgue measure on $\mathcal{C}([0,1])$ and on the two cones $\mathcal{C}_0((0,1])$ and $\mathcal{C}_0([0,1))$.

Idea of Proof

(i) This is essentially contained in [49]. For simplicity let us assume the trace τ_A is extremal, so that the weak closure of the GNS representation of A is a finite injective factor. We therefore obtain a unital *-homomorphism $A \to \mathcal{R} \subset \mathcal{R}^\omega$ which picks up the trace τ_A when composed with the canonical trace on $\mathcal{R}^\omega$. By Kaplansky's density theorem $\mathcal{Q}_\omega$ surjects onto $\mathcal{R}^\omega$, when dividing out the trace kernel ideal $\{x \in \mathcal{Q}_\omega \mid \tau_{\mathcal{Q}_\omega}(x^*x) = 0\} \lhd \mathcal{Q}_\omega$. By the Choi–Effros lifting theorem, there is a c.p.c. lift from A to $\mathcal{Q}_\omega$. The 'curvature' of this map (the defect of it being multiplicative) then lies in the trace kernel ideal of $\mathcal{Q}_\omega$, and one can use a quasicentral approximate unit in conjunction with a reindexing argument to replace it by a c.p.c. order zero lift. (Alternatively, one can use Kirchberg's ε-test from [31] in place of reindexing.) This order zero map corresponds to a *-homomorphism Ψ defined on the cone over A which will have the right properties.

(ii) Find a *-homomorphism $\lambda : \mathcal{C}_0((0,1]) \to \mathcal{Q}$ such that $\tau_{\mathcal{Q}} \circ \lambda = \tau_{\mathrm{Lebesgue}}$. Next, find a unital copy of $\mathcal{Q}$ in $\mathcal{Q}_\omega \cap \Psi(\mathcal{C}_0((0,1]) \otimes A)'$. We compose this inclusion with λ and tensor with Ψ to obtain a *-homomorphism

$$\widetilde{\Psi} : \mathcal{C}_0((0,1]) \otimes \mathcal{C}_0((0,1]) \otimes A \to \mathcal{Q}_\omega.$$

Since $\mathcal{C}_0((0,1])$ is the universal C*-algebra generated by a positive contraction, the assignment $\mathrm{id}_{(0,1]} \otimes a \mapsto \mathrm{id}_{(0,1]} \otimes \mathrm{id}_{(0,1]} \otimes a$ induces a *-homomorphism; we define $\acute{\Phi}$ to be the composition with $\widetilde{\Psi}$.

Next observe that the two cones in $\mathcal{Q}_\omega$ generated by the elements $\acute{\Phi}(\mathrm{id}_{(0,1]} \otimes 1_A)$ and $1_{\mathcal{Q}_\omega} - \acute{\Phi}(\mathrm{id}_{(0,1]} \otimes 1_A)$ carry the same Cuntz semigroup information (which is determined by Lebesgue measure on the interval), and are therefore unitarily equivalent by [8] (and reindexing), i.e., $1_{\mathcal{Q}_\omega} - \acute{\Phi}(\mathrm{id}_{(0,1]} \otimes 1_A) = w^* \acute{\Phi}(\mathrm{id}_{(0,1]} \otimes 1_A)\, w$ for some unitary $w \in \mathcal{Q}_\omega$. Define $\grave{\Phi}$ to be the resulting

conjugate of $\acute{\Phi}$, so that $\grave{\Phi}((1-\mathrm{id}_{[0,1)})\otimes a) = w^* \acute{\Phi}(\mathrm{id}_{(0,1]}\otimes a)w$, $a \in A$. The map Θ is then fixed by these data since $\grave{\Phi}((1-\mathrm{id}_{[0,1)})\otimes 1_A)+\acute{\Phi}(\mathrm{id}_{(0,1]}\otimes 1_A) = 1_{\mathcal{Q}_\omega}$.

∎

6.3 Now we have produced two cones over A inside $\mathcal{Q}_\omega$; the scalar parts of these fit nicely together, but the A-valued components might be in general position. The task is to join them in order to find a c.p.c. map from $\mathcal{C}([0,1]) \otimes A$ to (matrices over) $\mathcal{Q}_\omega$ which is either exactly or at least approximately multiplicative. We wish to establish this connection by comparing the two restrictions to the suspension over A,

$$\acute{\Lambda} := \acute{\Phi}|_{\mathcal{C}_0((0,1))\otimes A} : \mathcal{C}_0((0,1)) \otimes A \to \mathcal{Q}_\omega$$

and

$$\grave{\Lambda} := \grave{\Phi}|_{\mathcal{C}_0((0,1))\otimes A} : \mathcal{C}_0((0,1)) \otimes A \to \mathcal{Q}_\omega.$$

Here's what *would* make this work. It's not quite going to, but it is a blueprint of the actual proof, and it isolates the necessary ingredients.

Lemma *In the setting above, suppose there is a unitary $u \in \mathcal{Q}_\omega$ such that*

$$\grave{\Lambda}(\,.\,) = u\acute{\Lambda}(\,.\,)\,u^*. \tag{6.1}$$

Then, there is a *-homomorphism

$$\overline{\Phi} : A \to M_2(\mathcal{Q}_\omega)$$

such that

$$(\mathrm{tr}_{M_2} \otimes \tau_{\mathcal{Q}_\omega}) \circ \overline{\Phi} = \frac{1}{2} \cdot \tau_A.$$

In particular, the unital *-homomorphism

$$\widetilde{\Phi} : A \xrightarrow{\overline{\Phi}} \overline{\Phi}(1_A)\, M_2(\mathcal{Q}_\omega)\, \overline{\Phi}(1_A) \cong \mathcal{Q}_\omega$$

shows that the trace τ_A is quasidiagonal.

Proof We write down the map $\overline{\Phi}$ explicitly. Define a partition of unity of piecewise linear positive continuous functions $h_0, h_{1/2}, h_1$ on the interval so that h_0 equals 1 at 0, and is 0 on $[1/4, 1]$; h_1 is just h_0 reflected at $1/2$. Consider a continuous rotation $R \in M_2(\mathcal{C}([0,1]))$ with

$$R|_{[0,1/4]} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad R|_{[3/4,1]} \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let $\Theta^{(2)} : M_2(\mathcal{C}([0,1])) \to M_2(\mathcal{Q}_\omega)$ denote the amplification of Θ to 2×2 matrices.

We may then define a c.p. map by setting

$$\overline{\Phi}(a) := \begin{pmatrix} \grave{\Phi}(h_0 \otimes a) & 0 \\ 0 & 0 \end{pmatrix}$$
$$+ \begin{pmatrix} 1 & 0 \\ 0 & u^* \end{pmatrix} \Theta^{(2)}(R) \begin{pmatrix} \grave{\Lambda}(h_{1/2} \otimes a) & 0 \\ 0 & 0 \end{pmatrix} \Theta^{(2)}(R^*) \begin{pmatrix} 1 & 0 \\ 0 & u \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & 0 \\ 0 & \acute{\Phi}(h_1 \otimes a) \end{pmatrix}$$

for $a \in A$; it is not hard to check that $\overline{\Phi}$ is in fact multiplicative and picks up half of the trace τ_A as claimed in the lemma.

For the last statement note that $\overline{\Phi}$ is unital when regarded as a *-homomorphism to the hereditary C*-subalgebra generated by its image, $\overline{\Phi}(1_A)\, M_2(\mathcal{Q}_\omega)\, \overline{\Phi}(1_A)$, which is isomorphic to $\mathcal{Q}_\omega$ since $\mathcal{Q}$ is self-similar; cf. Remark 2.3. Under this identification the traces $2 \cdot (\mathrm{tr}_{M_2} \otimes \tau_{\mathcal{Q}_\omega})|_{\overline{\Phi}(1_A)\, M_2(\mathcal{Q}_\omega)\, \overline{\Phi}(1_A)}$ and $\tau_{\mathcal{Q}_\omega}$ agree since $\mathcal{Q}_\omega$ is monotracial by [37, Lemma 4.7], so that $\tau_A = \tau_{\mathcal{Q}_\omega} \circ \widetilde{\Phi}$ is quasidiagonal by Proposition 2.5. ∎

6.4 *Remarks*

(i) If one only had an approximate version of (6.1) the same argument would yield an approximately multiplicative c.p.c. map $\overline{\Phi}$; after reindexing this would still be good enough to prove quasidiagonality.

(ii) It is natural to ask whether the use of 2×2 matrices is really essential here. One could certainly hide the matrices by rotating and compressing everything into the upper left corner—but that's a red herring since one cannot necessarily force the resulting map to be unital. The reason is that the method above allows only limited control over K-theory, and one cannot guarantee that $\overline{\Phi}(1_A)$ is Murray–von Neumann equivalent to $e_{11} \otimes 1_{\mathcal{Q}_\omega}$ (of course the two agree tracially, but that's not enough in ultrapowers, even of UHF algebras).

6.5 In general, unitary equivalence of the two suspensions as in (6.1) seems too much to ask for—and the same goes for approximate versions. On the other hand, it's not completely outrageous either; for example, it is not too hard to see that when A happens to be strongly self-absorbing then the converse of Lemma 6.3 holds, i.e., unitary equivalence of the two suspensions is implied by quasidiagonality. More can be said using [5], but whether this kind of unitary equivalence is a necessary condition for quasidiagonality in complete generality is not clear, and we don't have means to check it directly. The way around this is the stable uniqueness machinery as introduced by Lin in [33], then refined by Dadarlat–Eilers in [11] and since often used and further refined by Elliott, Gong, Lin, Niu, and others.

6.6 Let us revisit Lemma 6.3 and replace the critical hypothesis (6.1) by a weaker one (still not quite weak enough for us to confirm it in sufficient generality, but now almost within reach):

$$\exists\, n \in \mathbb{N},\ u, v \in \mathcal{U}(M_{n+1}(\mathcal{Q}_\omega)) \text{ such that}$$

$$\grave{\Lambda} \oplus \grave{\Lambda}^{\oplus n} = u\Big(\acute{\Lambda} \oplus \grave{\Lambda}^{\oplus n}\Big)u^* \quad \text{and} \quad \grave{\Lambda} \oplus \acute{\Lambda}^{\oplus n} = v\Big(\acute{\Lambda} \oplus \acute{\Lambda}^{\oplus n}\Big)v^*. \tag{6.2}$$

Then we can chop the interval which sits via Θ inside $\mathcal{Q}_\omega \cong M_{2n} \otimes \mathcal{Q}_\omega$ into small pieces and apply the idea of the proof of Lemma 6.3 $2n$ times along the interval (we have to switch from u to v halfway, which is why we have to use $2n$ intervals, not just n); diagrammatically we end up with the following picture; cf. [51, Figure 1]:

$$\begin{array}{ccccccccccccc}
\grave{\Lambda} & & \grave{\Lambda} & & \grave{\Lambda} & & \grave{\Lambda} & & \grave{\Lambda} & & \grave{\Lambda} & & \acute{\Lambda} \\
\grave{\Lambda} & & \grave{\Lambda} & & \grave{\Lambda} & & \grave{\Lambda} & & \grave{\Lambda} & & \acute{\Lambda} & \sim_v & \acute{\Lambda} \\
\grave{\Lambda} & & \grave{\Lambda} & & \grave{\Lambda} & & \grave{\Lambda} & & \acute{\Lambda} & \sim_v & \acute{\Lambda} & & \acute{\Lambda} \\
\vdots & & \vdots & & \vdots & & \vdots & \dots & \vdots & & \vdots & & \vdots \\
\grave{\Lambda} & & \grave{\Lambda} & & \grave{\Lambda} & \sim_u & \grave{\Lambda} & & \acute{\Lambda} & & \acute{\Lambda} & & \acute{\Lambda} \\
\grave{\Lambda} & & \grave{\Lambda} & \sim_u & \grave{\Lambda} & & \acute{\Lambda} & & \acute{\Lambda} & & \acute{\Lambda} & & \acute{\Lambda} \\
\grave{\Lambda} & \sim_u & \grave{\Lambda} & & \acute{\Lambda} & & \acute{\Lambda} & & \acute{\Lambda} & & \acute{\Lambda} & & \acute{\Lambda} \\
\grave{\Lambda} & & \acute{\Lambda} & & \acute{\Lambda} & & \acute{\Lambda} & & \acute{\Lambda} & & \acute{\Lambda} & & \acute{\Lambda}
\end{array}$$

This will produce a *-homomorphism

$$\overline{\Phi} : A \to M_2 \otimes M_{2n} \otimes \mathcal{Q}_\omega$$

in a similar way as in Lemma 6.3, which again entails quasidiagonality.

6.7 Just as in Remark 6.4(i), it would be enough to come up with an approximately multiplicative c.p.c. map $\overline{\Phi}$, which would follow from an approximate version of (6.2). The latter is very close to the conclusion of Theorem 3.1, with $\grave{\Lambda}$ and $\acute{\Lambda}$ in place of φ and ψ, respectively, and also with $\grave{\Lambda}$ and $\acute{\Lambda}$ in place of ι. However, there is a catch: The maps in the diagrammatic chart of 6.6 are in fact not the original maps $\grave{\Lambda}$ or $\acute{\Lambda}$; rather, they are restrictions of those maps to small subintervals of $(0, 1)$. This makes a difference, since it means that the maps depend on the number of intervals, hence on n, which in Theorem 3.1 in turn depends on the maps—and the whole affair becomes circular! Luckily, there is a backdoor: In the controlled stable uniqueness theorem 3.6 the number n does not depend on the actual maps; it only depends (except for $\mathcal{G}$ and δ, of course) on the control function Δ which is tied to the Lebesgue measure on the interval via the prescribed trace and the map Θ. The price for this additional control is the UCT hypothesis in Theorem 5.1.

7 Some Open Problems

7.1 Of course the main problems in the context of this paper are the UCT problem and the quasidiagonality question in its various versions as discussed in Sect. 4. These are expected to be hard; the problems listed below aim to highlight their interplay and to break them up into smaller bits and pieces which will hopefully be easier to attack.

7.2 Question Are there formal implications between the versions of the quasidiagonality question from Sect. 4? In other words, can we prove any of the implications [$\mathbf{QDQ}_{\text{infinite s.s.a.}}$ holds] $\Longleftrightarrow$ [$\mathbf{QDQ}_{\text{finite s.s.a.}}$ holds] $\Longrightarrow$ [$\mathbf{QDQ}_{\text{simple},1}$ holds] $\Longrightarrow$ [$\mathbf{QDQ}_{\text{simple}}$ holds] $\Longrightarrow$ [**QDQ** holds]?

7.3 By Corollary 5.2, the UCT implies quasidiagonality under suitable conditions, and one can ask under which hypotheses there is a converse. This is also interesting for special cases:

Questions Does every quasidiagonal strongly self-absorbing C^*-algebra satisfy the UCT? What about strongly self-absorbing C^*-subalgebras of $\mathcal{Q} \otimes \mathcal{O}_\infty$? Or unital, simple, nuclear and monotracial C^*-subalgebras of $\mathcal{Z}$?

7.4 Kirchberg has reduced the UCT problem to the simple case; even more, the problem is equivalent to the question whether $\mathcal{O}_2$ is the only unital Kirchberg algebra with trivial K-theory (see [29, 2.17]). As discussed in 4.8, for strongly self-absorbing C^*-algebras the answer is known. From this point of view the following does not seem likely, but I still think it is worth asking.

Question Can the UCT problem be reduced to the strongly self-absorbing case?

7.5 It was shown in [53] that the K-theory of a strongly self-absorbing C^*-algebra satisfying the UCT has to agree with the K-theory of one of the known strongly self-absorbing examples. However, the proof really only requires the formally weaker Künneth Theorem for tensor products (see [48]), and one may ask whether even this can be made redundant, or whether there are at least some restrictions on the possible K-groups. For example, Dadarlat pointed out that for a quasidiagonal strongly self-absorbing $\mathcal{D}$, $K_1(\mathcal{D})$ cannot have an infinite cyclic subgroup (again by the Künneth Theorem and since in this case $\mathcal{D} \otimes \mathcal{Q} \cong \mathcal{Q}$).

Questions If $\mathcal{D}$ is a strongly self-absorbing C^*-algebra, does $K_1(\mathcal{D})$ have to be trivial? Does $K_*(\mathcal{D})$ have to be torsion free?

7.6 It is a classical question when a C^*-algebra is isomorphic to its opposite. Whenever one expects classification by K-theory data the answer should be positive, and it certainly is for strongly self-absorbing C^*-algebras with UCT; see

Corollary 5.6 (note that the opposite of a strongly self-absorbing C^*-algebra is again strongly self-absorbing).

Question Is a strongly self-absorbing $\mathcal{D}$ isomorphic to its opposite $\mathcal{D}^{\mathrm{op}}$?

7.7 The following stems essentially from [6].

Questions For a separable unital C^*-algebra A, do the quasidiagonal traces form a face? If, in addition, A is quasidiagonal, are all traces quasidiagonal? Do nuclearity of A or amenability of the traces make a difference?

Together with a result from [6], [51, Corollary 6.1] yields a positive answer to the second question when also assuming nuclearity and the UCT.

7.8 In both [39] and [51], quasidiagonality of amenable group C^*-algebras is derived abstractly from classification techniques—but at this point there is no way to construct quasidiagonalising finite-dimensional subspaces of $\ell^2(G)$ explicitly.

Question Is there a group theoretic / dynamic proof of Rosenberg's conjecture?

7.9 C^*-algebras of amenable groups are almost never simple—but they have simple quotients, and one may ask when these are classifiable. There is by now a range of very convincing results along these lines; cf. [14, 15].

Question When are simple quotients of amenable group C^*-algebras classifiable? When can one at least show $\mathcal{Z}$-stability?

7.10 In a similar vein, one can look at topological dynamical systems, where free and minimal actions typically yield simple C^*-crossed products. These algebras tend to be nuclear provided the groups—or at least their actions—are amenable; cf. [1, 54]. We know from [20] that one cannot expect regularity in general, and that conditions on the dimension of the underlying space (or again the action) are essential to guarantee $\mathcal{Z}$-stability or finite nuclear dimension of the crossed product. Recent results of Kerr, however, together with the tiling result of [13], suggest that we might be only a stone's throw away from an answer to the following:

Question For free minimal actions of countable discrete amenable groups on Cantor sets, are the crossed product C^*-algebras classifiable?

The setup is shockingly general: free minimal Cantor actions of amenable groups! So how are we even entitled to ask this? Quasidiagonality of the crossed product is given by our Theorem 5.1 in connection with [55], which verifies the UCT. $\mathcal{Z}$-stability seems now within reach with Kerr's techniques on tiling (based on [13]) together with Archey and Phillips' large subalgebra approach [2] or, alternatively, using the idea of dynamic dimension and dynamic $\mathcal{Z}$-stability as defined by the

author (yet unpublished, but closely related to the notion of Rokhlin dimension from [26]). From here only finite nuclear dimension of the crossed product would be missing to arrive at classifiability (by [17] via [62]; for a slightly more direct approach in the uniquely ergodic case see [49]). In the case when the ergodic measures form a compact space, finite nuclear dimension follows from $\mathcal{Z}$-stability by [5].
Here is an even more general—though not necessarily more daring—layout.

Questions For free minimal actions of countable discrete amenable groups on finite dimensional spaces, are the crossed product C*-algebras classifiable? What about amenable actions of countable discrete groups?

The following rigidity question was beautifully answered for Cantor minimal $\mathbb{Z}$-actions in [21] in terms of strong orbit equivalence. In the situation of amenable group actions, it seems much more speculative, and one can only expect a less complete answer. If one is prepared to go beyond the context of amenable group actions, Popa's rigidity theory for von Neumann algebras (cf. [43]) is extremely encouraging—but on the C*-algebra side one would have to change the game completely and develop most of the technology from scratch.

Question To what extent are topological dynamical systems determined by their associated C*-algebras?

References

1. Claire Anantharaman-Delaroche. Amenability and exactness for dynamical systems and their C*-algebras. *Trans. Amer. Math. Soc.*, 354:4153–4178, 2002.
2. Dawn Archey and N. Christopher Phillips. Permanence of stable rank one for centrally large subalgebras and crossed products by minimal homeomorphisms. arXiv:1505.00725, 2015.
3. Bruce Blackadar. K*-theory for operator algebras*, volume 5 of *Mathematical Sciences Research Institute Publications*. Cambridge University Press, Cambridge, second edition, 1998.
4. Bruce Blackadar and Eberhard Kirchberg. Generalized inductive limits of finite-dimensional C*-algebras. *Math. Ann.*, 307(3):343–380, 1997.
5. Joan Bosa, Nathanial P. Brown, Yasuhiko Sato, Aaron Tikuisis, Stuart White, and Wilhelm Winter. Covering dimension of C*-algebras and 2-coloured classification. *Mem. Amer. Math. Soc.*, to appear. arXiv:1506.03974v2, 2015.
6. Nathanial P. Brown. Invariant means and finite representation theory of C*-algebras. *Mem. Amer. Math. Soc.*, 184(865):viii+105, 2006.
7. Man Duen Choi and Edward G. Effros. The completely positive lifting problem for C*-algebras. *Ann. of Math. (2)*, 104(3):585–609, 1976.
8. Alin Ciuperca and George A. Elliott. A remark on invariants for C*-algebras of stable rank one. *Int. Math. Res. Not. IMRN*, 2008(5):Art. ID rnm 158, 33 pp., 2008.
9. Alain Connes. Classification of injective factors. Cases II_1, II_∞, III_λ, $\lambda \neq 1$. *Ann. of Math. (2)*, 104(1):73–115, 1976.

10. Marius Dadarlat. Some remarks on the universal coefficient theorem in KK-theory. In *Operator algebras and mathematical physics (Constanţa)*, pages 65–74, 2003.
11. Marius Dadarlat and Søren Eilers. On the classification of nuclear C^*-algebras. *Proc. London Math. Soc. (3)*, 85(1):168–210, 2002.
12. Marius Dadarlat and Wilhelm Winter. On the KK-theory of strongly self-absorbing C^*-algebras. *Math. Scand.*, 104(1):95–107, 2009.
13. Tomasz Downarowicz, Dawid Huczek, and Guohua Zhang. Tilings of amenable groups. arXiv:1502.02413, 2015.
14. Siegfried Echterhoff, Wolfgang Lück, N. Christopher Phillips, and Samuel Walters. The structure of crossed products of irrational rotation algebras by finite subgroups of $SL_2(\mathbb{Z})$. *J. Reine Angew. Math.*, 639:173–221, 2010.
15. Caleb Eckhardt and Elizabeth Gillaspy. Irreducible representations of nilpotent groups generate classifiable C^*-algebras. *Münster J. Math.*, to appear. arXiv:1510.05469, 2015.
16. Edward G. Effros and Jonathan Rosenberg. C^*-algebras with approximately inner flip. *Pacific J. Math.*, 77(2):417–443, 1978.
17. George A. Elliott, Guihua Gong, Huaxin Lin, and Zhuang Niu. On the classification of simple amenable C^*-algebras with finite decomposition rank, II. arXiv:1507.03437v2, 2015.
18. George A. Elliott and Andrew S. Toms. Regularity properties in the classification program for separable amenable C^*-algebras. *Bull. Amer. Math. Soc. (N.S.)*, 45(2):229–245, 2008.
19. James Gabe. Quasidiagonal traces on exact C^*-algebras. arXiv:1511.02760, 2015.
20. Julien Giol and David Kerr. Subshifts and perforation. *J. Reine Angew. Math.*, 639:107–119, 2010.
21. Thierry Giordano, Ian F. Putnam, and Christian F. Skau. Topological orbit equivalence and C^*-crossed products. *J. Reine Angew. Math.*, 469:51–111, 1995.
22. Guihua Gong, Huaxin Lin, and Zhuang Niu. Classification of finite simple amenable $\mathcal{Z}$-stable C^*-algebras. arXiv:1501.00135v4, 2015.
23. Uffe Haagerup. A new proof of the equivalence of injectivity and hyperfiniteness for factors on a separable Hilbert space. *J. Funct. Anal.*, 62(2):160–201, 1985.
24. Don Hadwin. Strongly quasidiagonal C^*-algebras. *J. Operator Theory*, 18(1):3–18, 1987. With an appendix by Jonathan Rosenberg.
25. Paul. R. Halmos. Ten problems in Hilbert space. *Bull. Amer. Math. Soc.*, 76:887–933, 1970.
26. Ilan Hirshberg, Wilhelm Winter, and Joachim Zacharias. Rokhlin dimension and C^*-dynamics. *Comm. Math. Phys.*, 335:637–670, 2015.
27. Xinhui Jiang and Hongbing Su. On a simple unital projectionless C^*-algebra. *Amer. J. Math.*, 121:359–413, 1999.
28. Eberhard Kirchberg. Exact C^*-algebras, tensor products, and the classification of purely infinite algebras. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 943–954. Birkhäuser, Basel, 1995.
29. Eberhard Kirchberg. Central sequences in C^*-algebras and strongly purely infinite algebras. In *Operator Algebras: The Abel Symposium 2004*, volume 1 of *Abel Symp.*, pages 175–231. Springer, Berlin, 2006.
30. Eberhard Kirchberg and N. Christopher Phillips. Embedding of exact C^*-algebras in the Cuntz algebra $\mathcal{O}_2$. *J. Reine Angew. Math.*, 525:17–53, 2000.
31. Eberhard Kirchberg and Mikael Rørdam. Central sequence C^*-algebras and tensorial absorption of the Jiang–Su algebra. *J. Reine Angew. Math.*, 695:175–214, 2014.
32. Eberhard Kirchberg and Wilhelm Winter. Covering dimension and quasidiagonality. *Internat. J. Math.*, 15(1):63–85, 2004.
33. Huaxin Lin. Stable approximate unitary equivalence of homomorphisms. *J. Operator Theory*, 47(2):343–378, 2002.
34. Huaxin Lin. AF-embeddings of the crossed products of AH-algebras by finitely generated abelian groups. *Int. Math. Res. Pap. IMRP*, 2008(3):Art. ID rpn007, 67 pp., 2008.
35. Hiroki Matui and Yasuhiko Sato. Strict comparison and $\mathcal{Z}$-absorption of nuclear C^*-algebras. *Acta Math.*, 209(1):179–196, 2012.

36. Hiroki Matui and Yasuhiko Sato. Decomposition rank of UHF-absorbing C*-algebras. *Duke Math. J.*, 163(14):2687–2708, 2014.
37. Hiroki Matui and Yasuhiko Sato. $\mathcal{Z}$-stability of crossed products by strongly outer actions II. *Amer. J. Math.*, 136(6):1441–1496, 2014.
38. Dusa McDuff. Central sequence and the hyperfinite factor. *Proc. London Math. Soc. (3)*, 21:443–461, 1970.
39. Narutaka Ozawa, Mikael Rørdam, and Yasuhiko Sato. Elementary amenable groups are quasidiagonal. *Geom. Funct. Anal.*, 25(1):307–316, 2015.
40. Mihai V. Pimsner. Embedding some transformation group C^*-algebras into AF-algebras. *Ergodic Theory Dynam. Systems*, 3(4):613–626, 1983.
41. Sorin Popa. A short proof of "Injectivity Implies Hyperfiniteness" for finite von Neumann algebras. *J. Operator Theory*, 16:261–272, 1986.
42. Sorin Popa. On local finite-dimensional approximation of C*-algebras. *Pacific J. Math.*, 181(1):141–158, 1997.
43. Sorin Popa. Deformation and rigidity for group actions and von Neumann algebras. In *International Congress of Mathematicians. Vol. I*, pages 445–477. Eur. Math. Soc., Zürich, 2007.
44. Mikael Rørdam. Classification of nuclear, simple C*-algebras. In *Classification of nuclear* C*-*algebras. Entropy in operator algebras*, volume 126 of *Encyclopaedia Math. Sci.*, pages 1–145. Springer, Berlin, 2002.
45. Mikael Rørdam. A simple C*-algebra with a finite and an infinite projection. *Acta Math.*, 191:109–142, 2003.
46. Mikael Rørdam. The stable and the real rank of $\mathcal{Z}$-absorbing C*-algebras. *Internat. J. Math.*, 15(10):1065–1084, 2004.
47. Mikael Rørdam and Wilhelm Winter. The Jiang–Su algebra revisited. *J. Reine Angew. Math.*, 642:129–155, 2010.
48. Jonathan Rosenberg and Claude Schochet. The Künneth theorem and the universal coefficient theorem for Kasparov's generalized K-functor. *Duke Math. J.*, 55(2):431–474, 1987.
49. Yasuhiko Sato, Stuart White, and Wilhelm Winter. Nuclear dimension and $\mathcal{Z}$-stability. *Invent. Math.*, 202:893–921, 2015.
50. Georges Skandalis. Une notion de nucléarité en K-théorie (d'après J. Cuntz). K-*Theory*, 1(6):549–573, 1988.
51. Aaron Tikuisis, Stuart White, and Wilhelm Winter. Quasidiagonality of nuclear C*-algebras. arXiv:1509.08318, 2015.
52. Andrew S. Toms. On the classification problem for nuclear C*-algebras. *Ann. of Math. (2)*, 167:1029–1044, 2008.
53. Andrew S. Toms and Wilhelm Winter. Strongly self-absorbing C*-algebras. *Trans. Amer. Math. Soc.*, 359(8):3999–4029, 2007.
54. Andrew S. Toms and Wilhelm Winter. Minimal dynamics and K-theoretic rigidity: Elliott's conjecture. *Geom. Funct. Anal.*, 23(1):467–481, 2013.
55. Jean-Louis Tu. La conjecture de Baum–Connes pour les feuilletages moyennables. K-*Theory*, 17(3):215–264, 1999.
56. Dan Voiculescu. A note on quasi-diagonal C*-algebras and homotopy. *Duke Math. J.*, 62(2):267–271, 1991.
57. Dan Voiculescu. Around quasidiagonal operators. *Integral Equations Operator Theory*, 17(1):137–149, 1993.
58. Dan Voiculescu. A non-commutative Weyl–von Neumann theorem. *Rev. Roum. Math. Pures et Appl.*, 97–113, 21.
59. Wilhelm Winter. Decomposition rank and $\mathcal{Z}$-stability. *Invent. Math.*, 179(2):229–301, 2010.
60. Wilhelm Winter. Strongly self-absorbing C*-algebras are $\mathcal{Z}$-stable. *J. Noncommut. Geom.*, 5(2):253–264, 2011.
61. Wilhelm Winter. Nuclear dimension and $\mathcal{Z}$-stability of pure C*-algebras. *Invent. Math.*, 187(2):259–342, 2012.

62. Wilhelm Winter. Classifying crossed product C^*-algebras. *Amer. J. Math.*, 138(3):793–820, 2016.
63. Wilhelm Winter and Joachim Zacharias. The nuclear dimension of C^*-algebras. *Adv. Math.*, 224(2):461–498, 2010.